Introduction to Quantum Metrology

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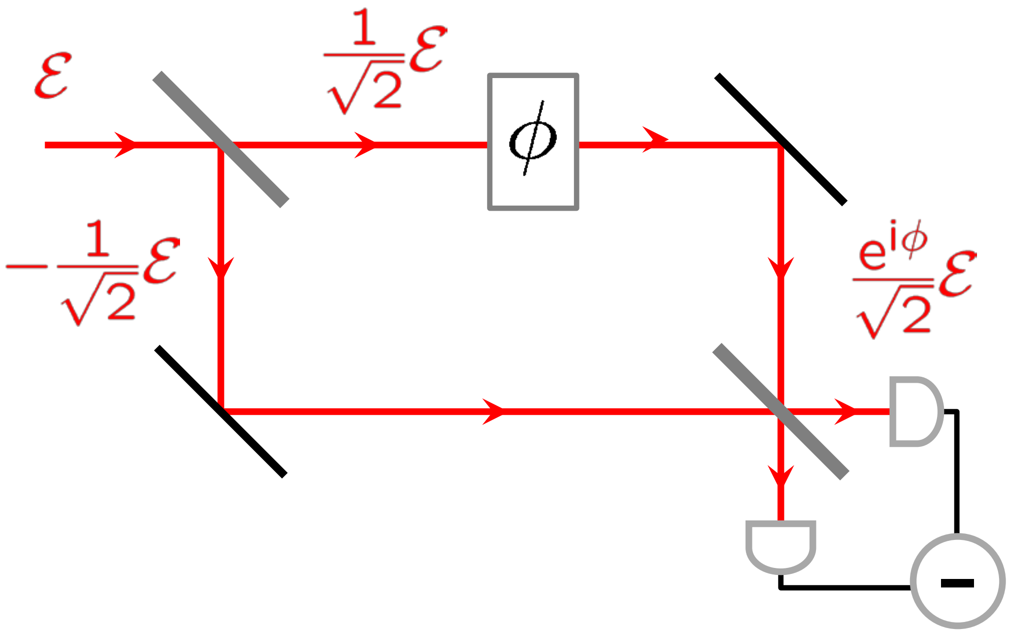
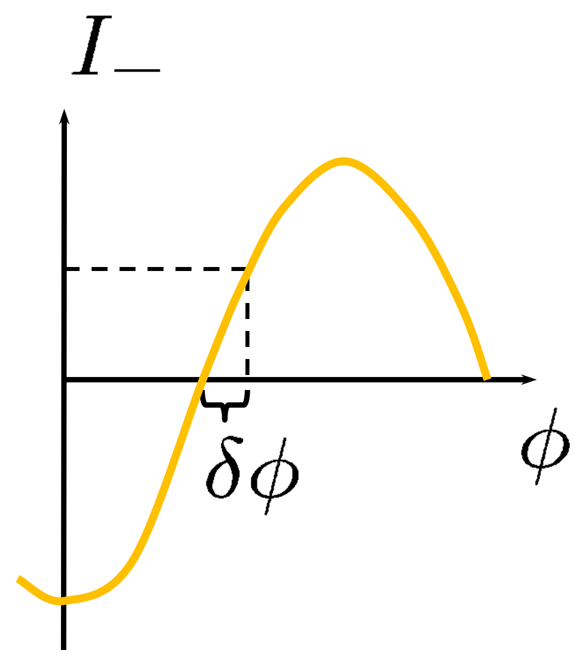
Summer School on Quantum and Nonlinear Optics
Sørup Herregaard
12.06.2015

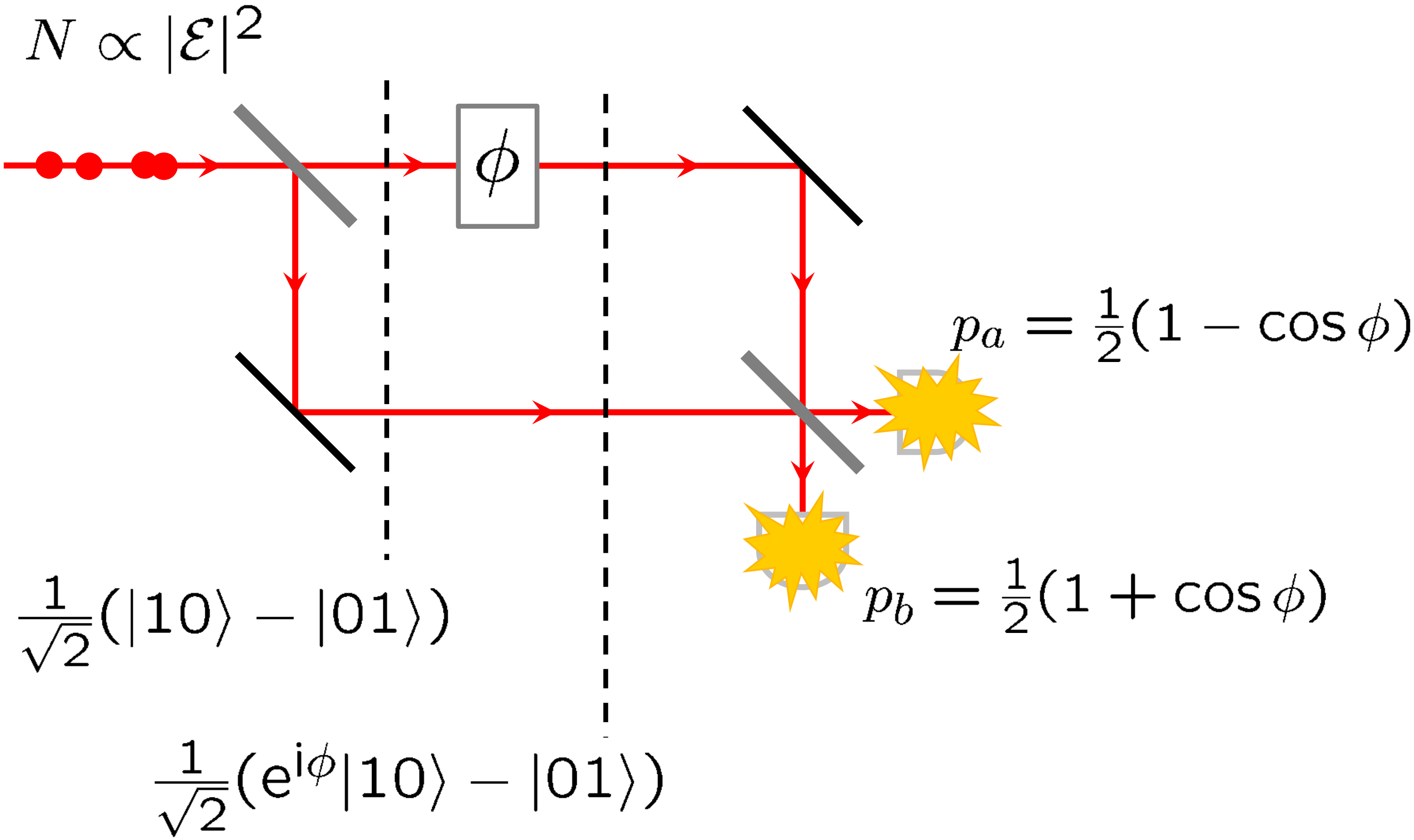
Phase measurement



$$I_{-} = -|\mathcal{E}|^2 \cos\phi$$



Quantum picture

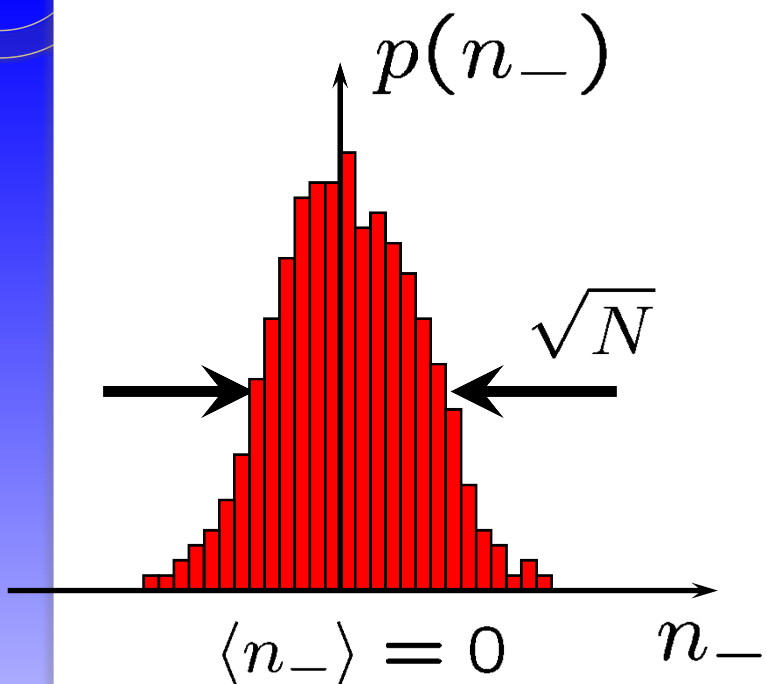


Photocount number difference:

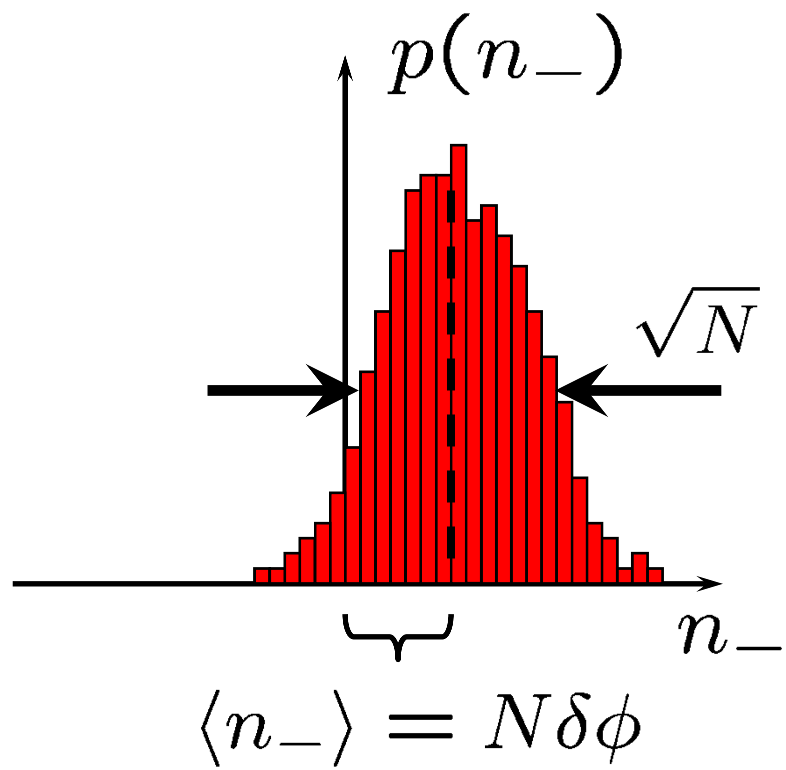
$$n_- = n_a - n_b$$

Shot noise

No phase shift $\delta\phi = 0$



Phase shift $\delta\phi \neq 0$



To identify a phase shift $N\delta\phi \gtrsim \sqrt{N}$

hence phase resolution $\delta\phi \gtrsim \frac{1}{\sqrt{N}}$

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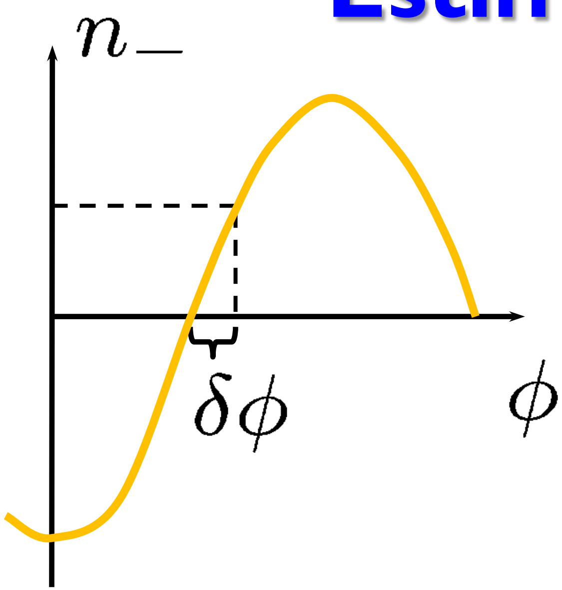
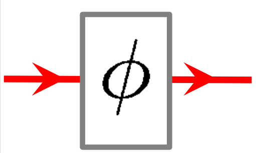
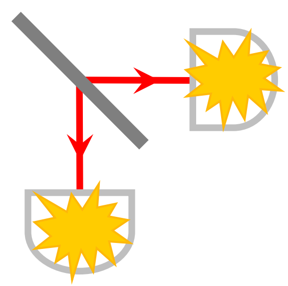
Estimation procedure



$$\Phi(n_a, n_b) = \frac{\pi}{2} + \frac{n_a - n_b}{N}$$

Actual value ϕ → $p(r|\phi)$ → Measurement result r → $\Phi(r)$ → Estimate $\tilde{\phi} = \Phi(r)$

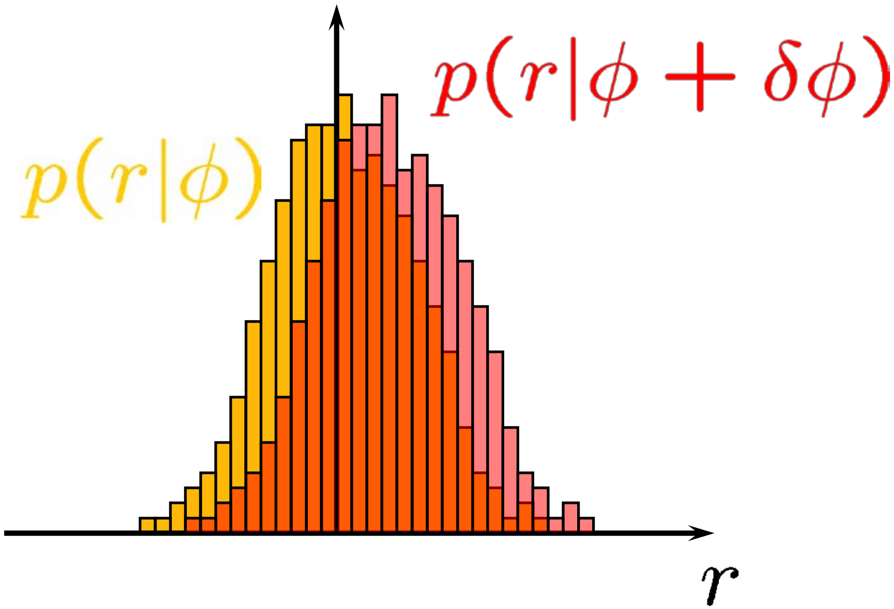




(n_a, n_b)

Fisher information

$$\mathsf{F}(\phi) = \sum_r p(r|\phi) \left(\frac{\partial}{\partial \phi} \ln p(r|\phi) \right)^2$$



Cramér-Rao bound:
for unbiased estimators

$$\Delta\tilde{\phi} \geq \frac{1}{\sqrt{\mathsf{F}(\phi)}}$$

Proof of the Cramér-Rao bound

Start from the Cauchy-Schwarz inequality:

$$\left(\sum_r A_r^2\right)\left(\sum_r B_r^2\right) \geq \left(\sum_r A_r B_r\right)^2$$

Take:

$$A_r = \sqrt{p(r|\phi)}[\Phi(r) - \phi]$$

$$B_r = \frac{1}{\sqrt{p(r|\phi)}}\frac{\partial}{\partial\phi}p(r|\phi)$$

and use on the right hand side the *unbiasedness* condition:

$$\sum_r p(r|\phi)\Phi(r) = \phi$$

Additivity

For statistically independent variables

$$p(r_a, r_b|\phi) = p(r_a|\phi)p(r_b|\phi)$$

Fisher information is additive:

$$\mathsf{F}(\phi) = \mathsf{F}_a(\phi) + \mathsf{F}_b(\phi)$$

One photon sent into the Mach-Zehnder interferometer

$$\mathsf{F}_1(\phi) = 1$$

Hence for N independently used photons $\mathsf{F}_N(\phi) = N$
and precision is bounded by *shot noise limit*:

$$\Delta\tilde{\phi} \geq \frac{1}{\sqrt{N}}$$

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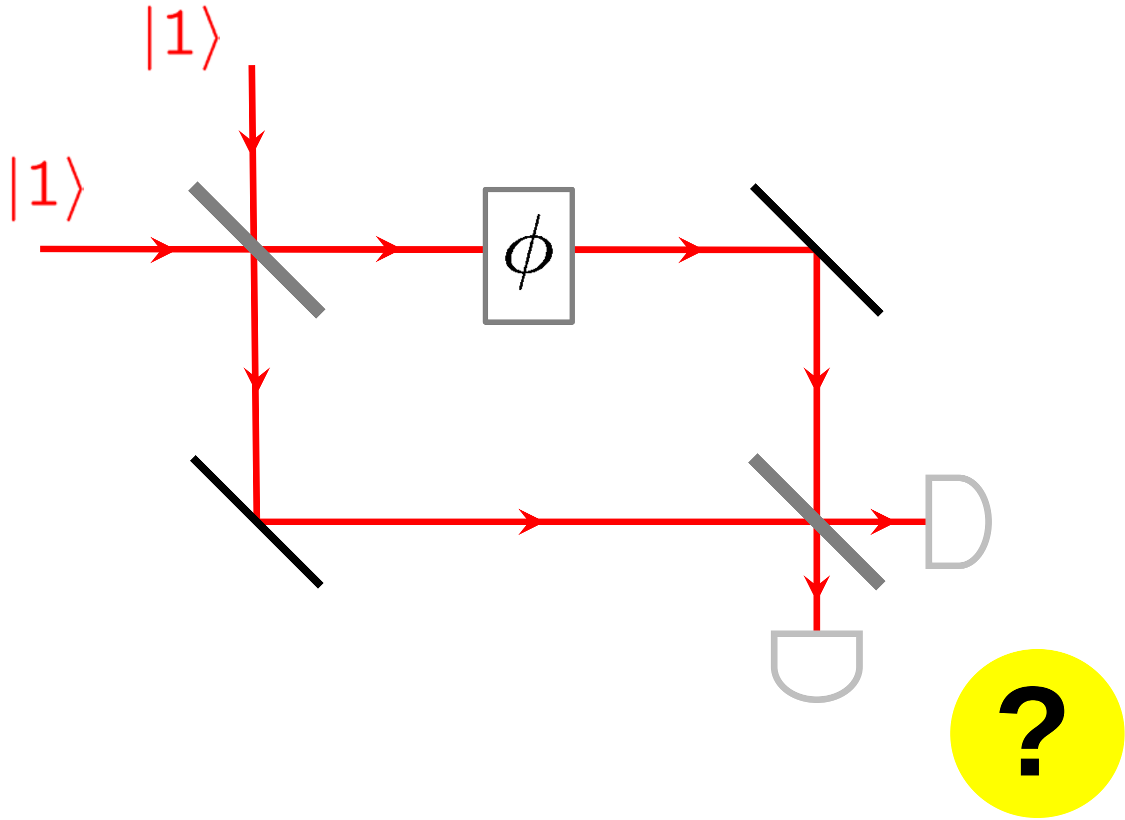
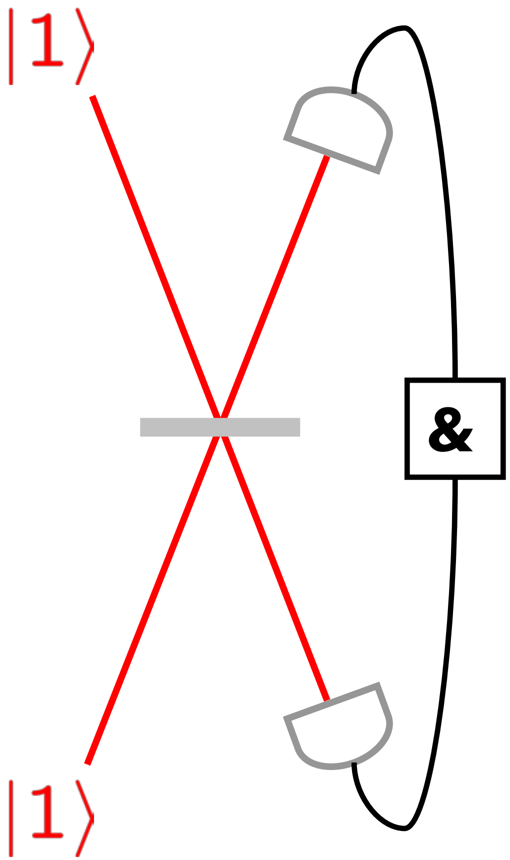
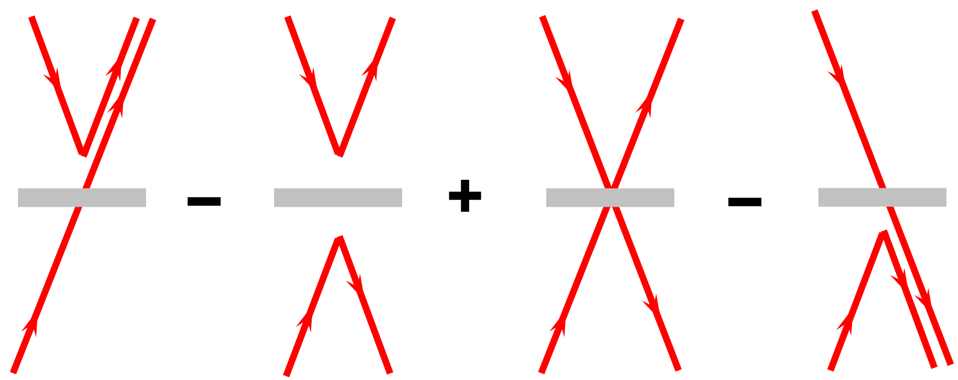
Interferometer with photon pairs



Two-photon interference



Probability amplitudes:



$$|11\rangle \rightarrow \frac{1}{\sqrt{2}}(|20\rangle - |02\rangle)$$

Only if photons are *indistinguishable!*

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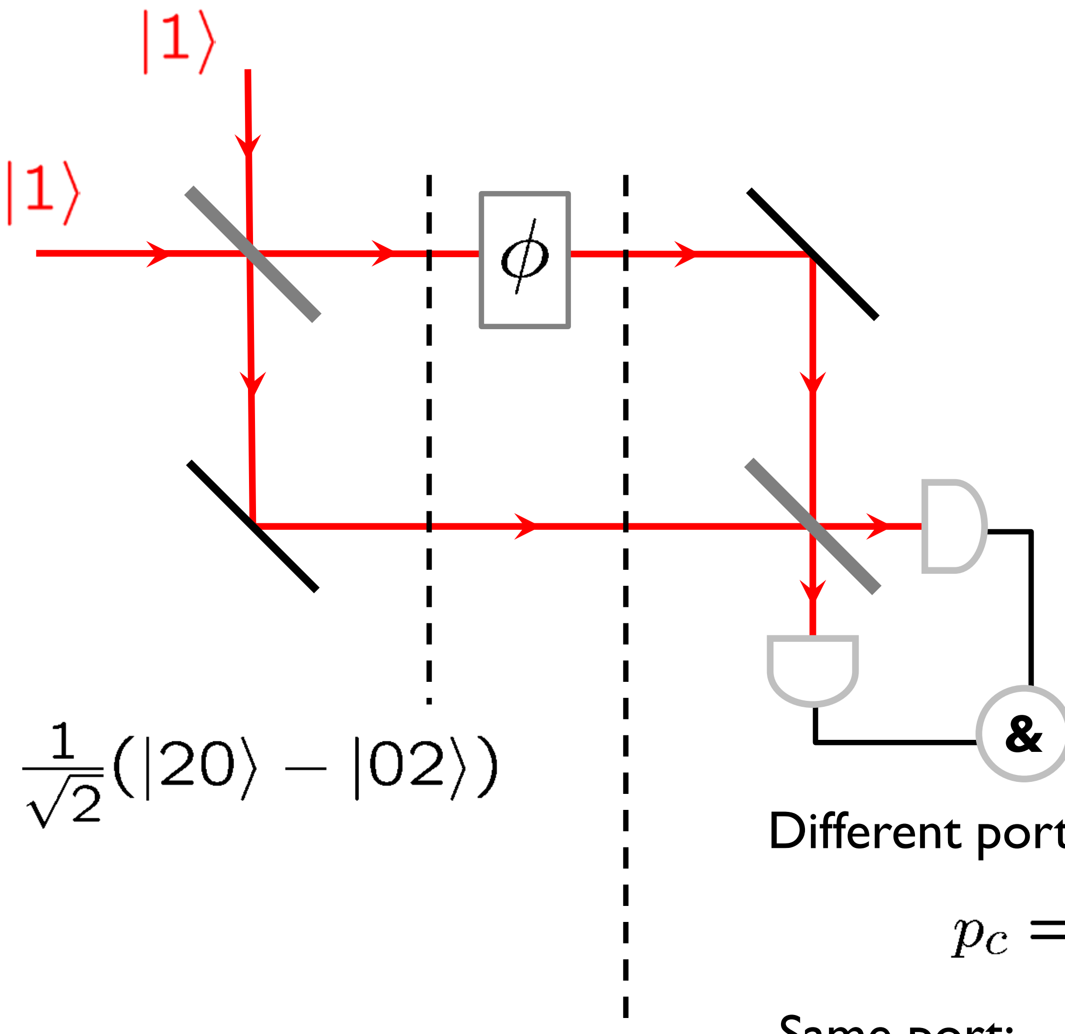
Two-photon interferometry



$$\frac{1}{\sqrt{2}}(|20\rangle - |02\rangle)$$

$$\frac{1}{\sqrt{2}}(e^{2i\phi}|20\rangle - |02\rangle)$$

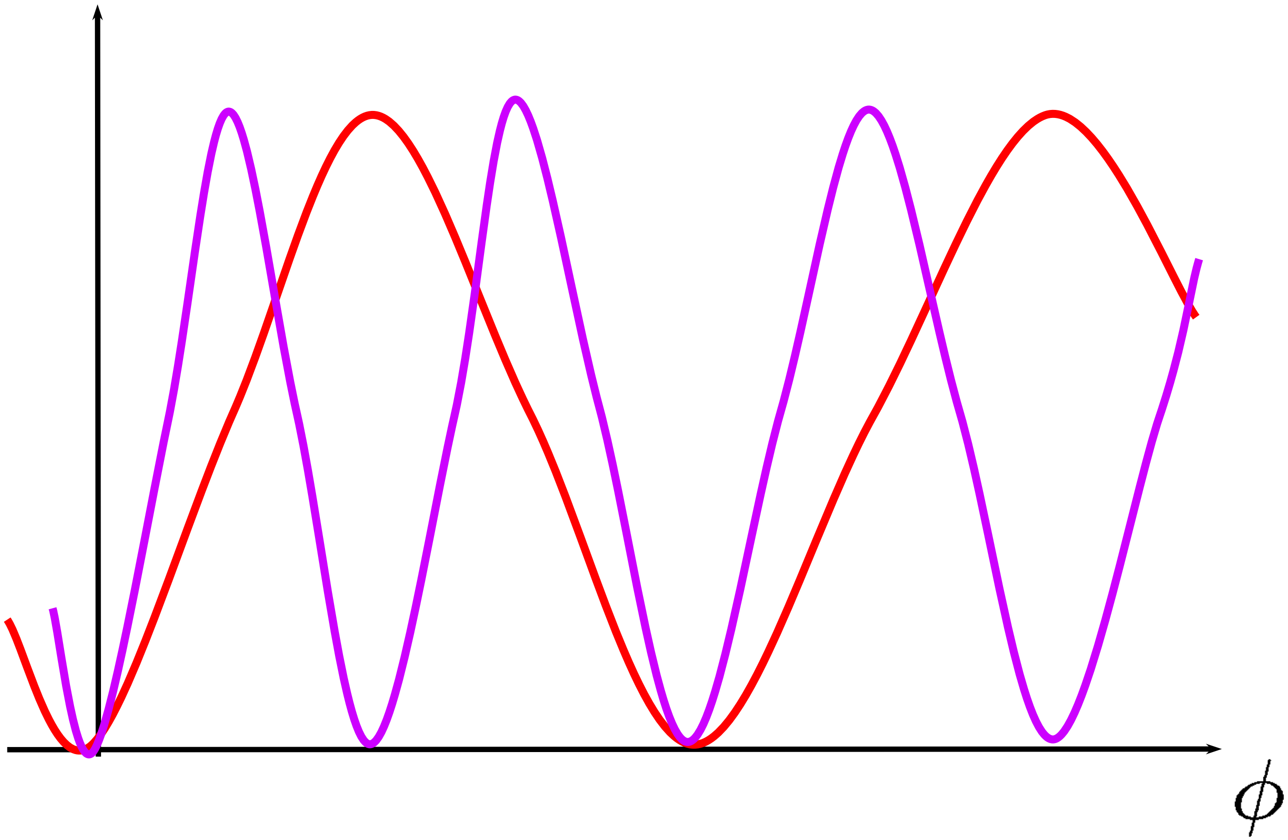
Different ports:

$$p_c = \frac{1}{2}(1 + \cos 2\phi)$$

Same port:

$$p_d = \frac{1}{2}(1 - \cos 2\phi)$$

Fringe spacing

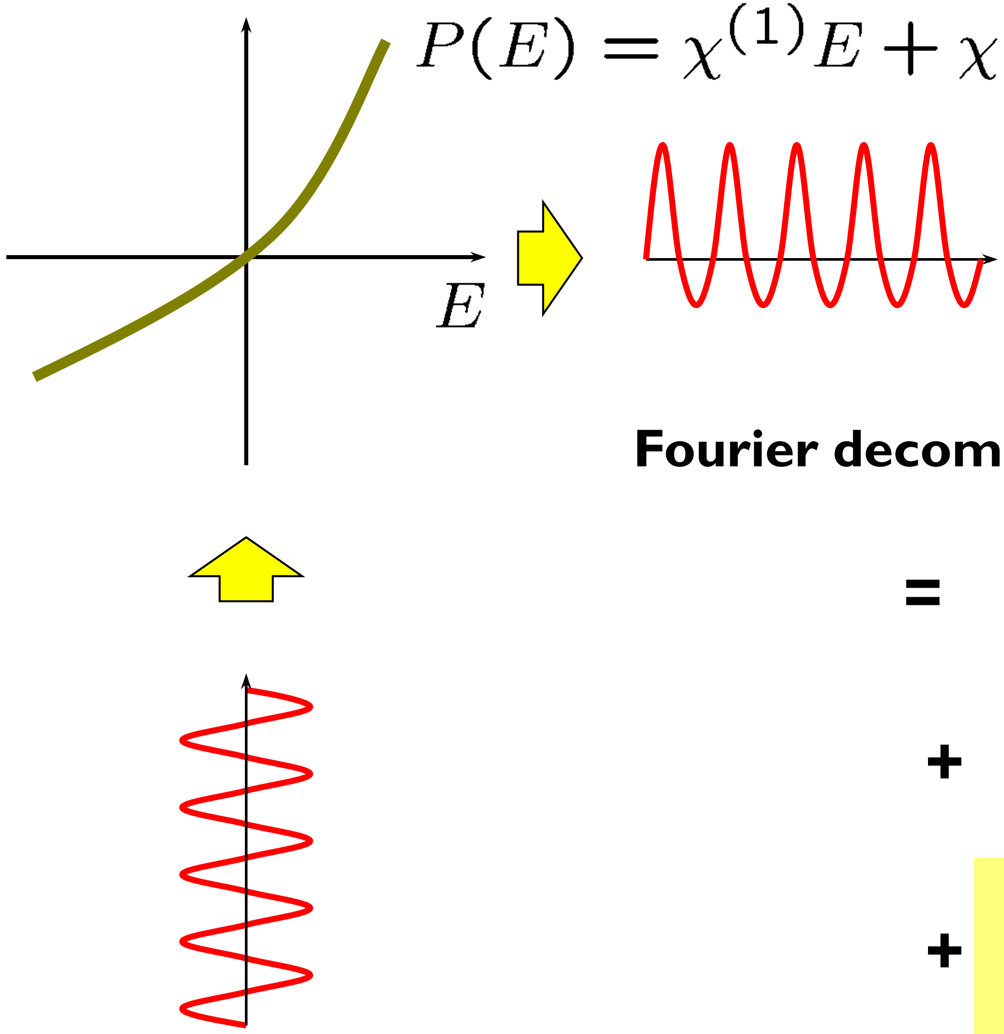


Two photons sent one-by one: $F = 2$

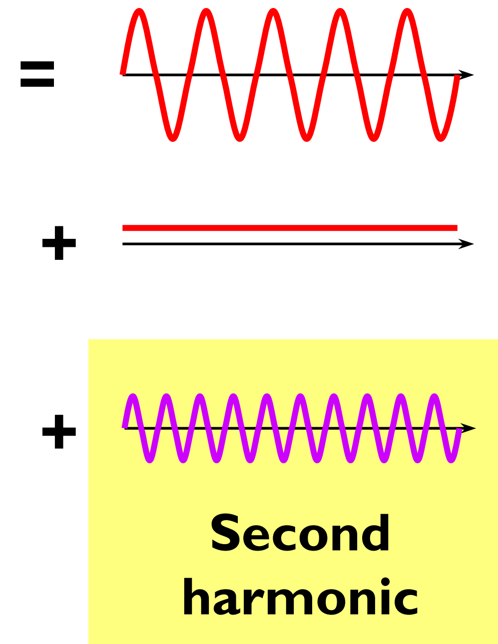
Two-photon interference: $F = 4$

Nonlinear susceptibility

$$P(E) = \chi^{(1)} E + \chi^{(2)} E^2 + \ldots$$

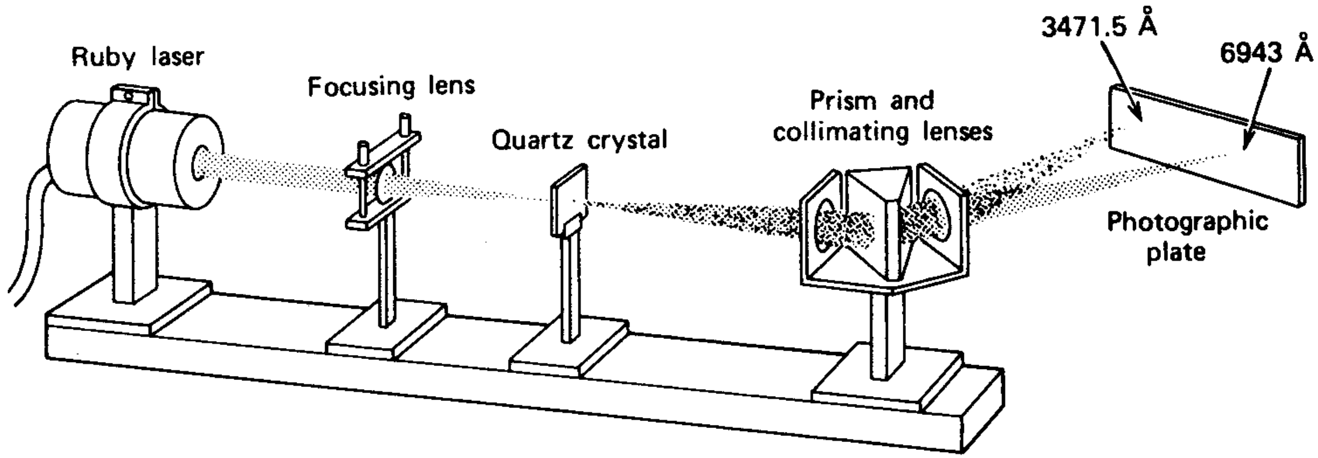


Fourier decomposition:



Second harmonic generation

P.A. Franken *et al*., Phys. Rev. Lett. **7**, 118 (1961)



VOLUME 7, NUMBER 4 PHYSICAL REVIEW LETTERS AUGUST 15, 1961

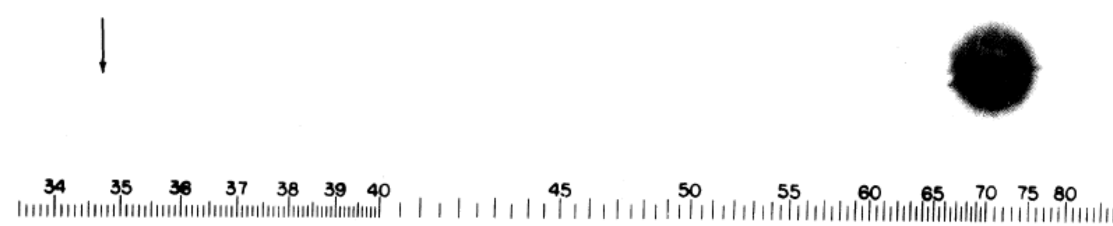


FIG. 1. A direct reproduction of the first plate in which there was an indication of second harmonic. The wavelength scale is in units of 100 A. The arrow at 3472 A indicates the small but dense image produced by the second harmonic. The image of the primary beam at 6943 A is very large due to halation.

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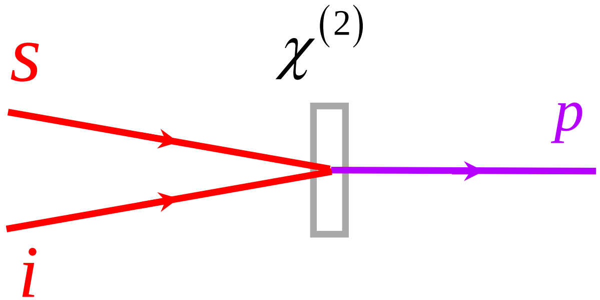
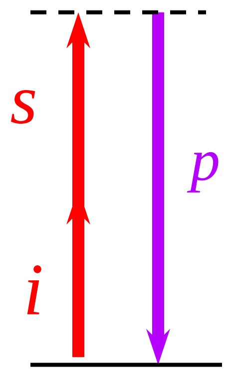
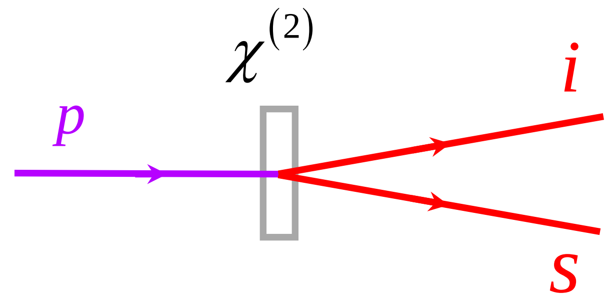
Three-wave mixing

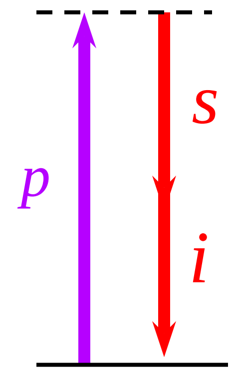
Sum frequency generation:



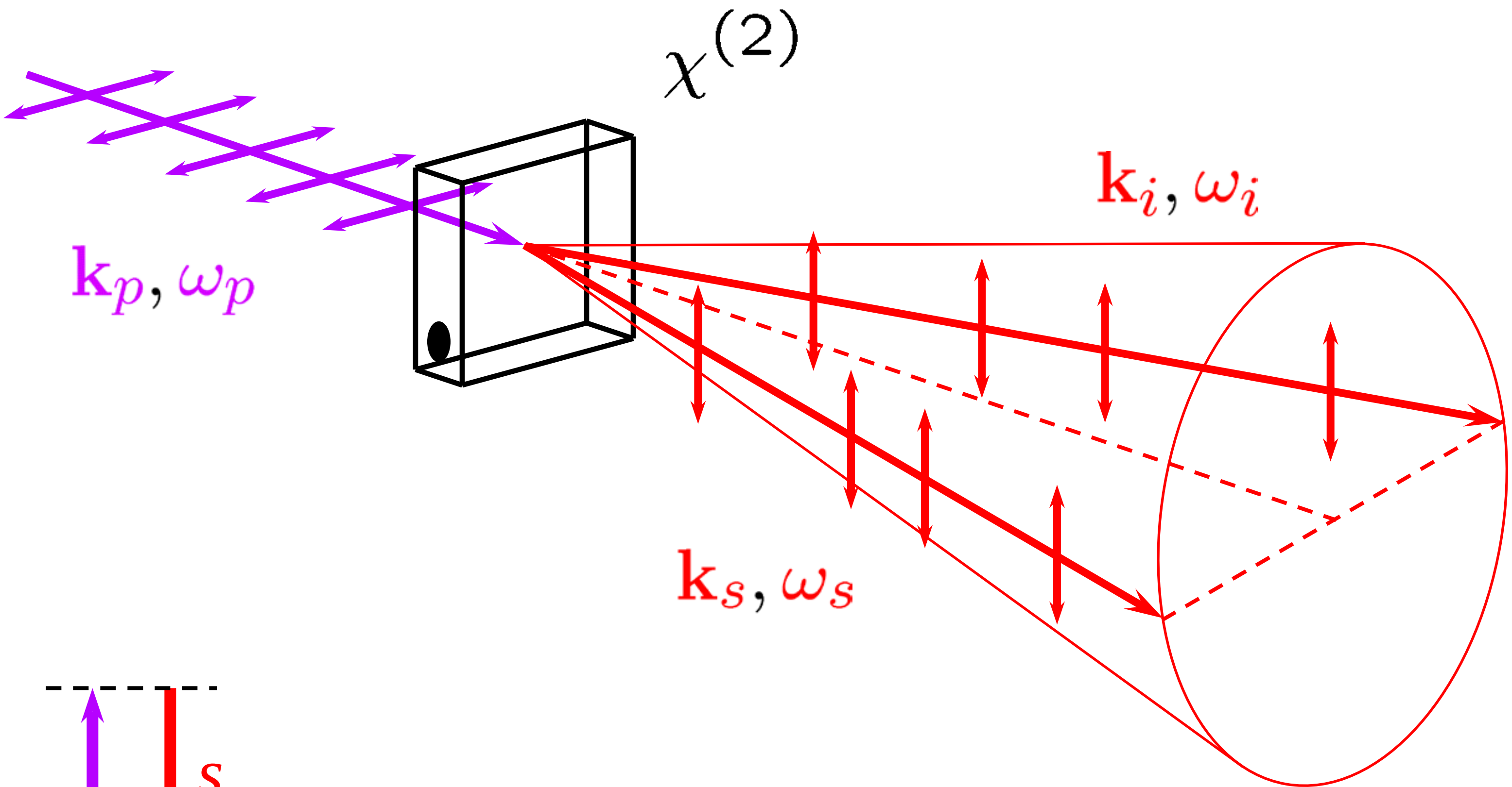


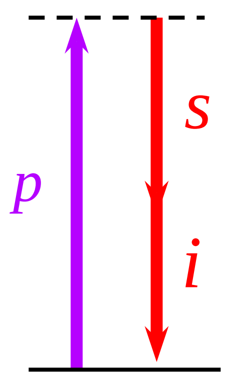
Parametric down-conversion:





Type-I process





Energy conservation: $\omega_p = \omega_s + \omega_i$

Momentum conservation: $\mathbf{k}_p \approx \mathbf{k}_s + \mathbf{k}_i$

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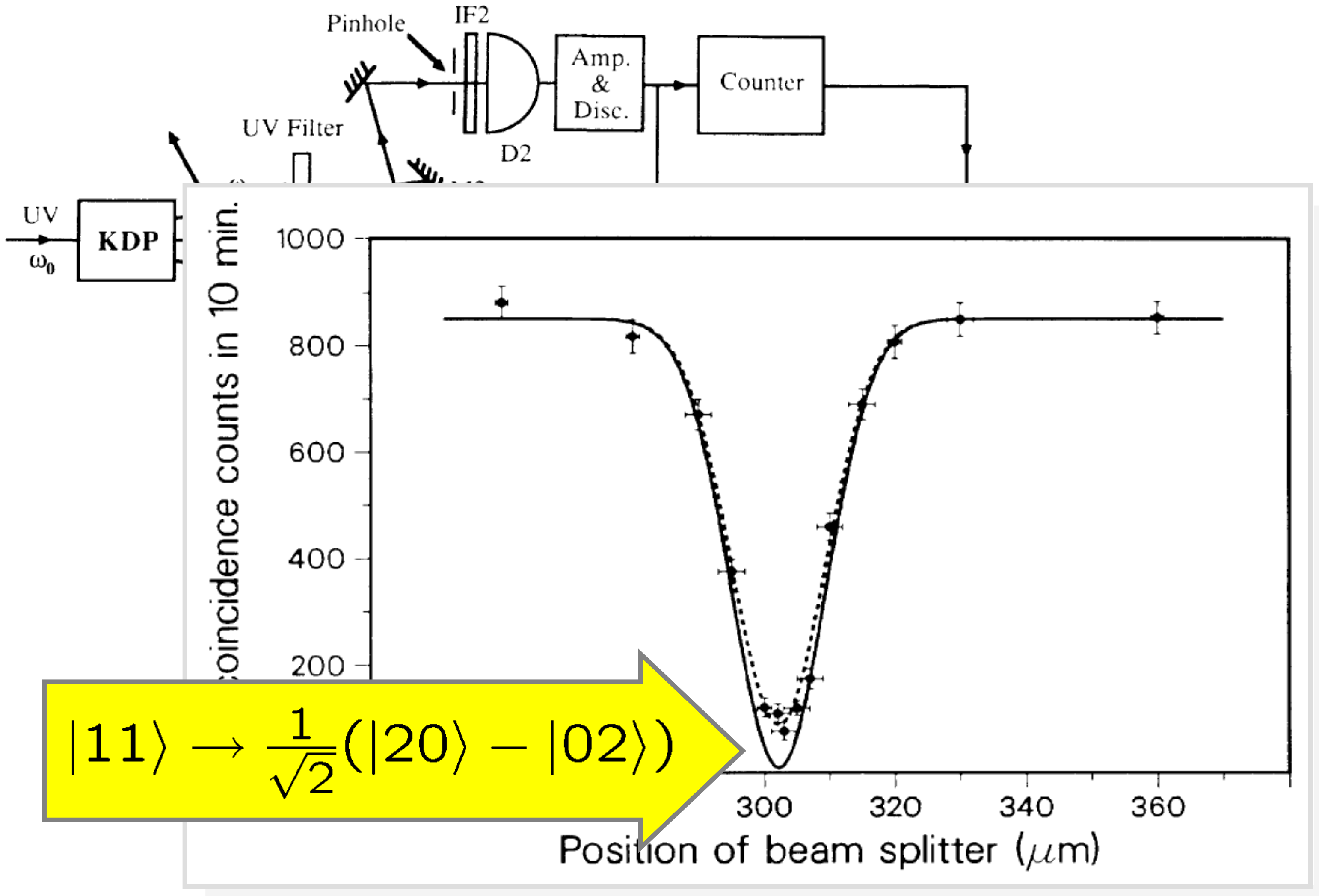
Two-photon interference

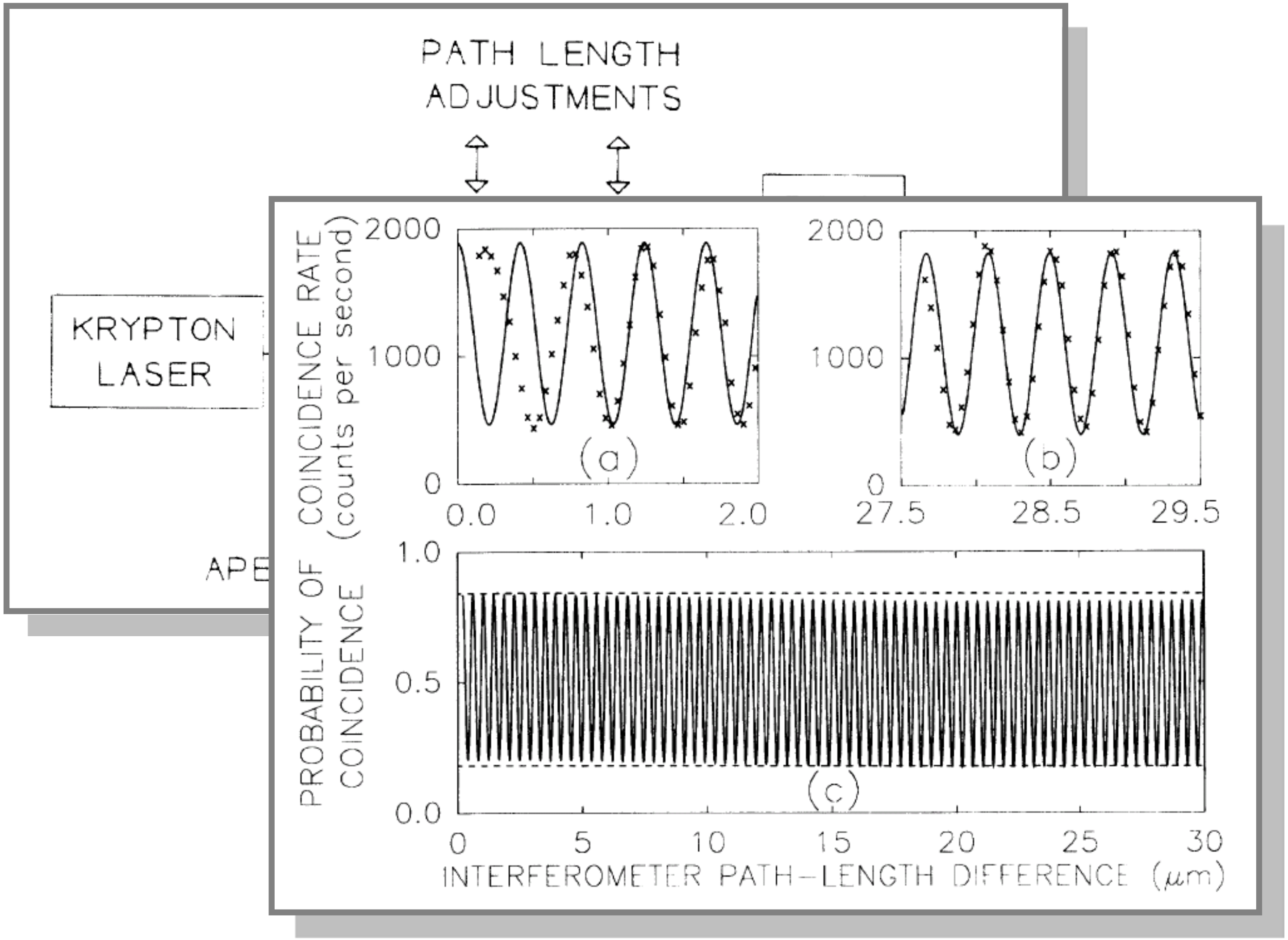


$$|11\rangle \rightarrow \frac{1}{\sqrt{2}}(|20\rangle - |02\rangle)$$

C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987)

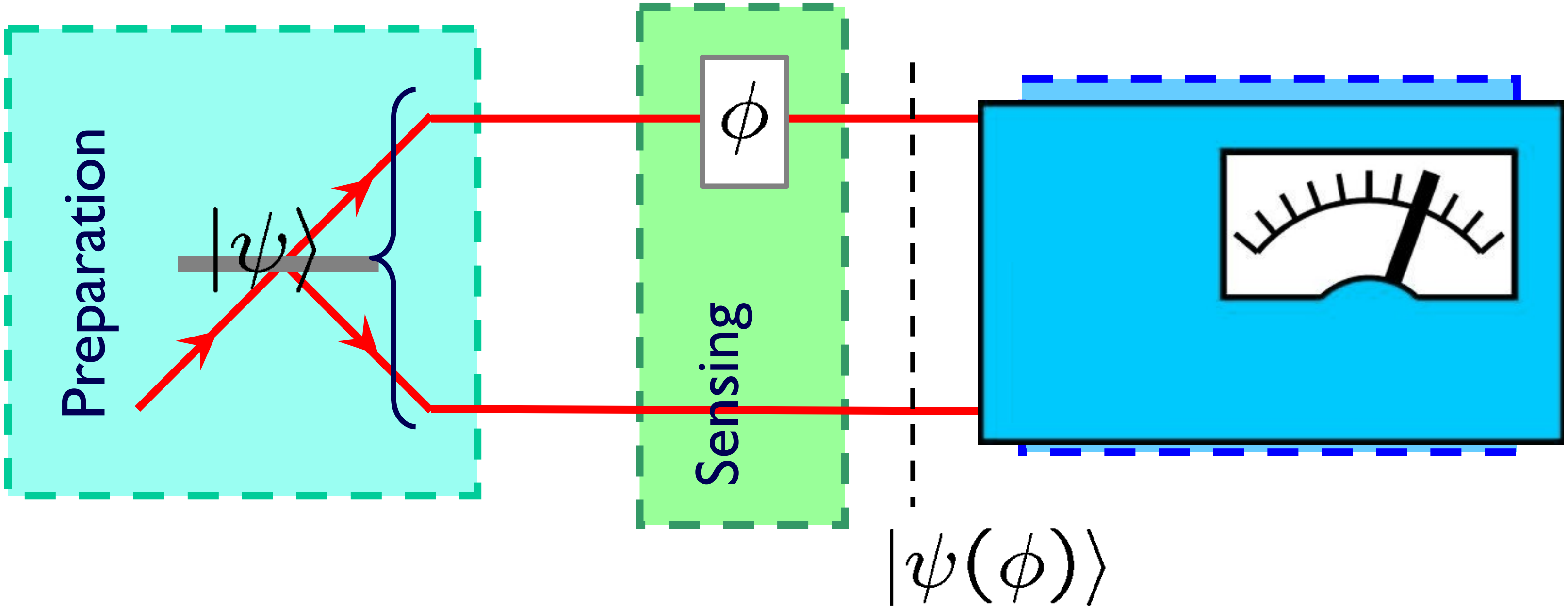
Experiment

J. G. Rarity *et al.,* Phys. Rev. Lett. **65**, 1348 (1990)

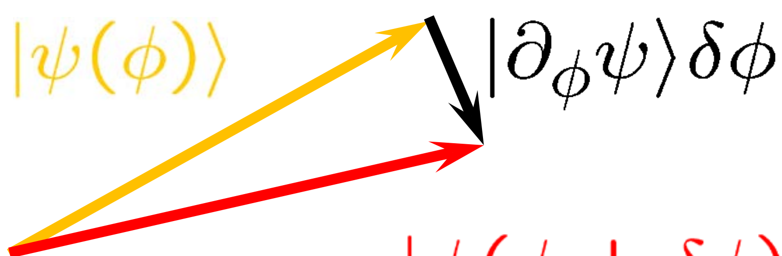


Photon wavelength 826.8 nm

General picture



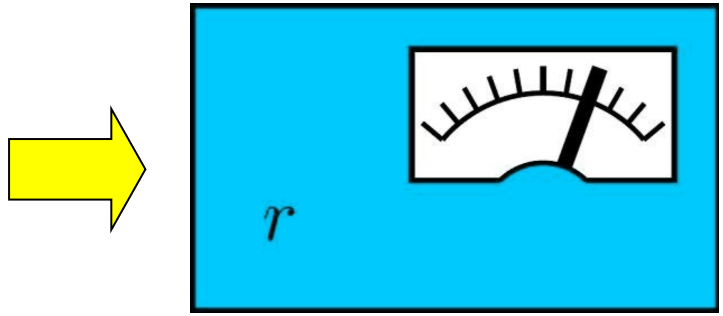
Local (inifinitesimal) phase estimation:



$$|\psi(\phi + \delta\phi)\rangle \approx |\psi(\phi)\rangle + |\partial_\phi \psi\rangle \delta\phi$$

Quantum Fisher information

$|\psi_\phi\rangle$



$$\hat{M}_r \geq 0$$
$$\sum_r \hat{M}_r = \hat{\mathbb{1}}$$

Probability of outcome *r*: $p(r|\phi) = \langle \psi_\phi | \hat{M}_r | \psi_\phi \rangle$

For any measurement

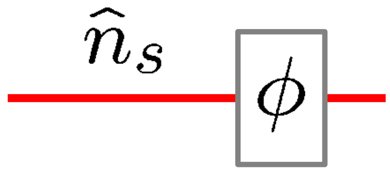
$$\mathsf{F}(\phi) \leq \mathsf{F}_Q(\phi)$$

where *quantum Fisher information*

$$\mathsf{F}_Q(\phi) = 4\Big(\langle \partial_\phi \psi | \partial_\phi \psi \rangle - |\langle \psi(\phi) | \partial_\phi \psi \rangle|^2\Big)$$

depends only on $|\psi(\phi)\rangle$

Uncertainty relation



Phase measurement:

$$|\psi(\phi)\rangle = \mathrm{e}^{\mathrm{i}\hat{n}_s\phi}|\psi\rangle$$

Explicit expression $F_Q(\phi) = 4(\Delta n_s)^2$ yields:

$$\Delta\tilde{\phi}\Delta n_s \geq \frac{1}{2}$$

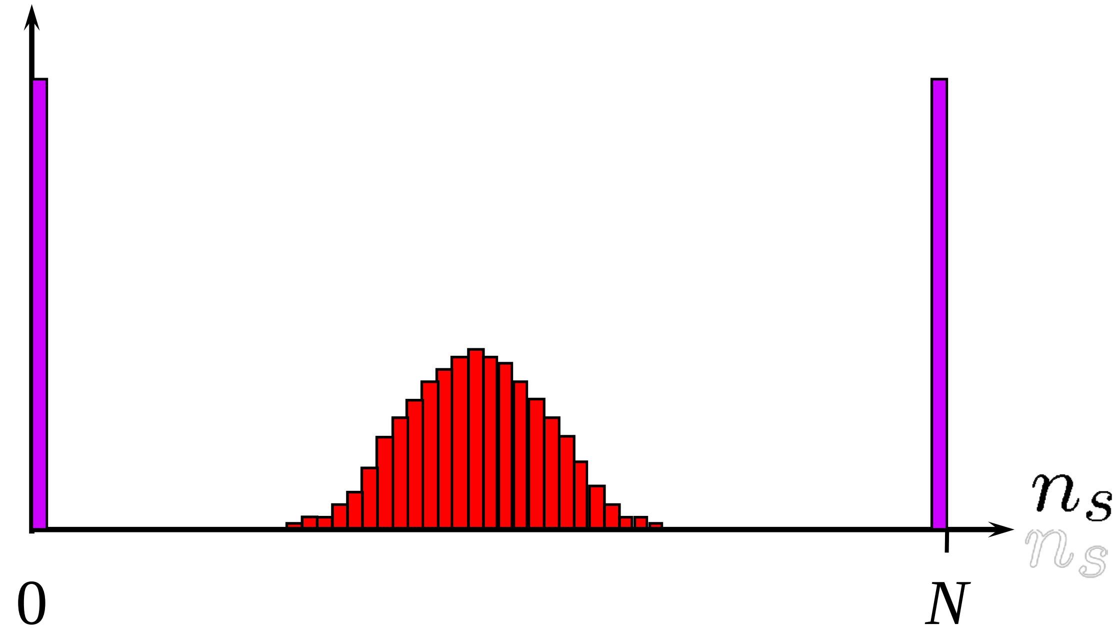
Δn_s – photon number uncertainty in the sensing arm

$\Delta\tilde{\phi}$ – precision of phase estimation

Task: maximize Δn_s for a fixed total number of N photons.

Optimal precision

Total N photons:



N independently used photons (shot noise limit):

$$\Delta\tilde{\phi} = \frac{1}{\sqrt{N}}$$

Maximum possible Δn_s defines the *Heisenberg limit*:

$$\Delta\tilde{\phi} = \frac{1}{N}$$

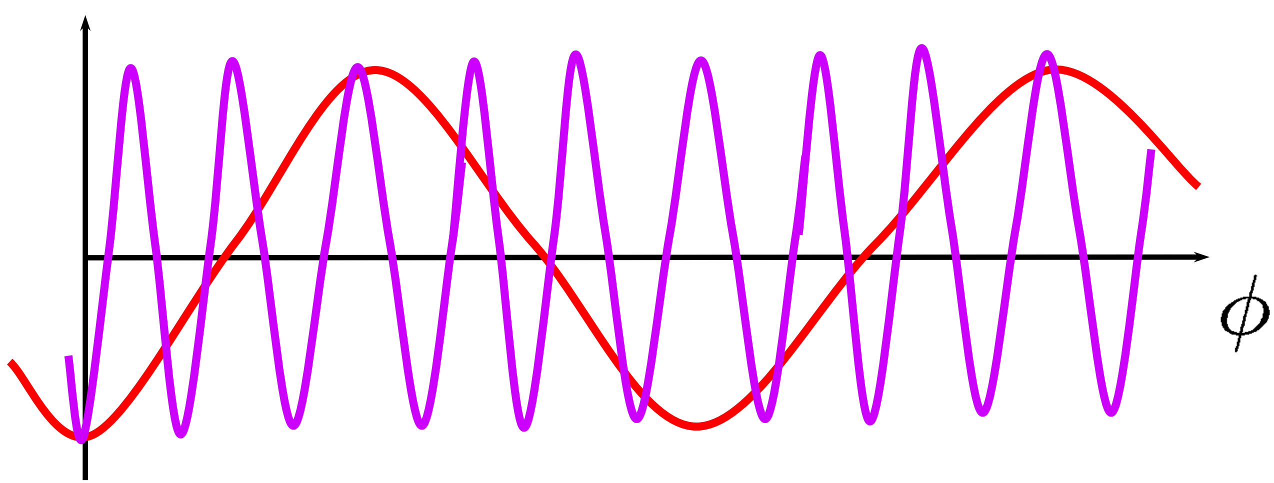
J. J. Bollinger *et al.*, Phys. Rev. A **54**, R4649 (1996).

N00N state

Optimal N photon state:

$$|\Psi_N\rangle = \frac{1}{\sqrt{2}}(|N0\rangle - |0N\rangle)$$

$$\longrightarrow \frac{1}{\sqrt{2}}(e^{iN\phi}|N0\rangle - |0N\rangle)$$



Review: V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004)

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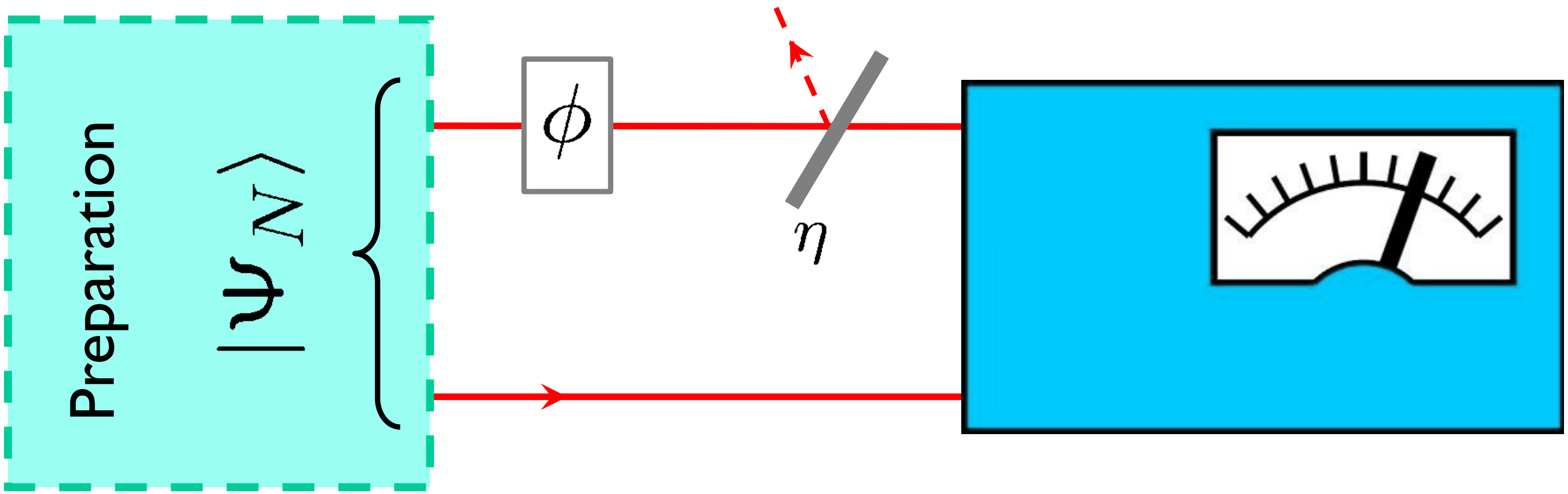
Losses



$$|\Psi_N(\phi)\rangle = \frac{1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}N\phi}|N0\rangle - |0N\rangle)$$

No photon lost: $\sqrt{\eta}^N \mathrm{e}^{\mathrm{i}N\phi}|N0\rangle - |0N\rangle$

One photon lost: $\mathrm{e}^{\mathrm{i}N\phi}|N-1,0\rangle$

More photons…

M.A. Rubin and S. Kaushik, Phys. Rev. A **75**, 053805 (2007)
G. Gilbert, M. Hamrick, Y.S. Weinstein, J. Opt. Soc. Am. B **25**, 1336 (2008)

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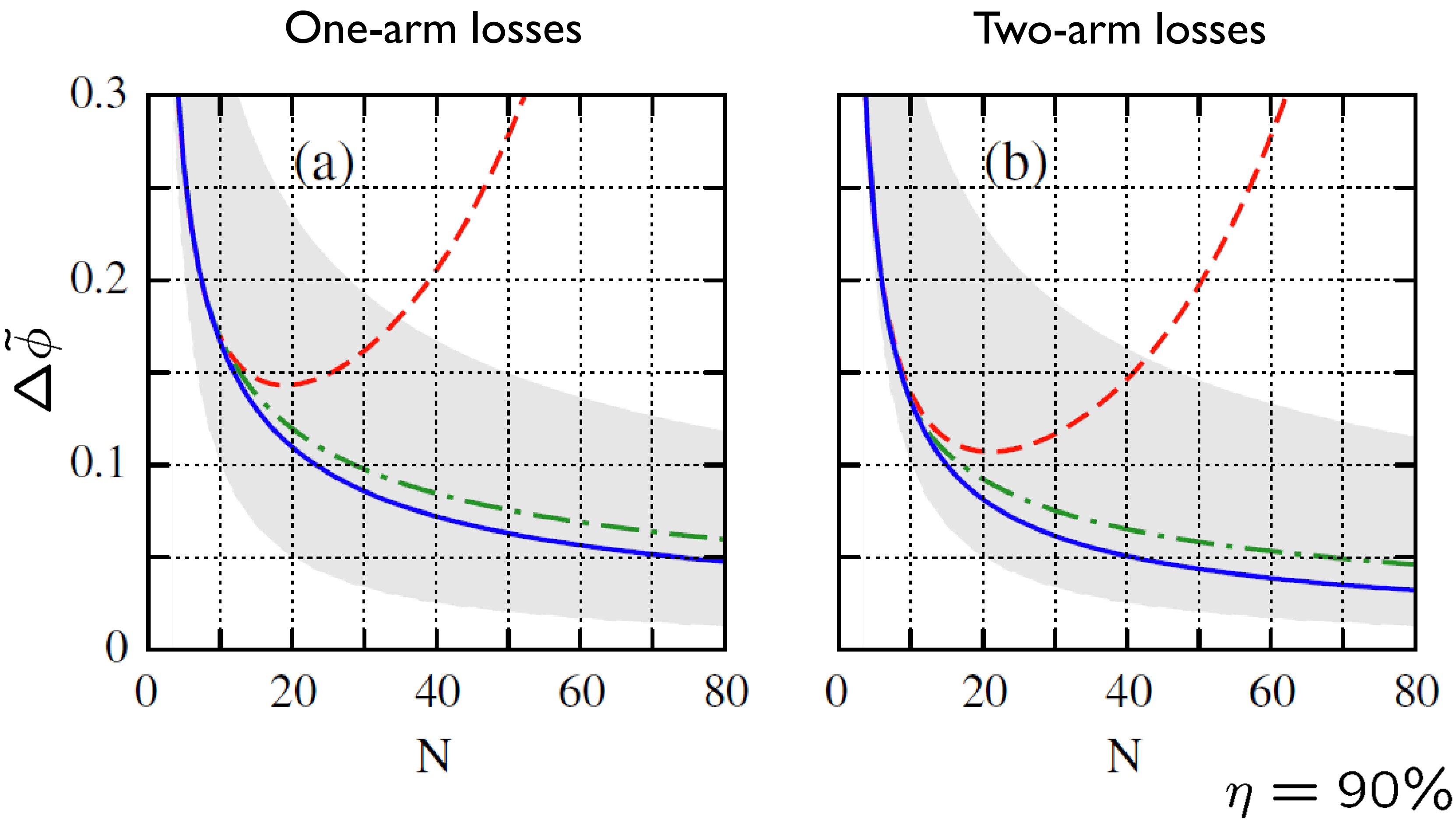
Attainable precision



Optimal

Chopped n00n

N00N state

U. Dorner, R. Demkowicz-Dobrzański *et al.*, Phys. Rev. Lett. **102**, 040403 (2009)

R. Demkowicz-Dobrzański, U. Dorner *et al.*, Phys. Rev. A **80**, 013825 (2009)

Two-photon states

$$|\psi\rangle = \alpha|20\rangle + \beta|11\rangle + \gamma|02\rangle$$

No photon lost:

$$\eta\alpha|20\rangle + \sqrt{\eta}\beta|11\rangle + \gamma|02\rangle$$

One photon lost:

$$\sqrt{2\eta(1-\eta)}\alpha|10\rangle + \sqrt{1-\eta}\beta|01\rangle$$

Two photons lost:

$$(1-\eta)\alpha|00\rangle$$

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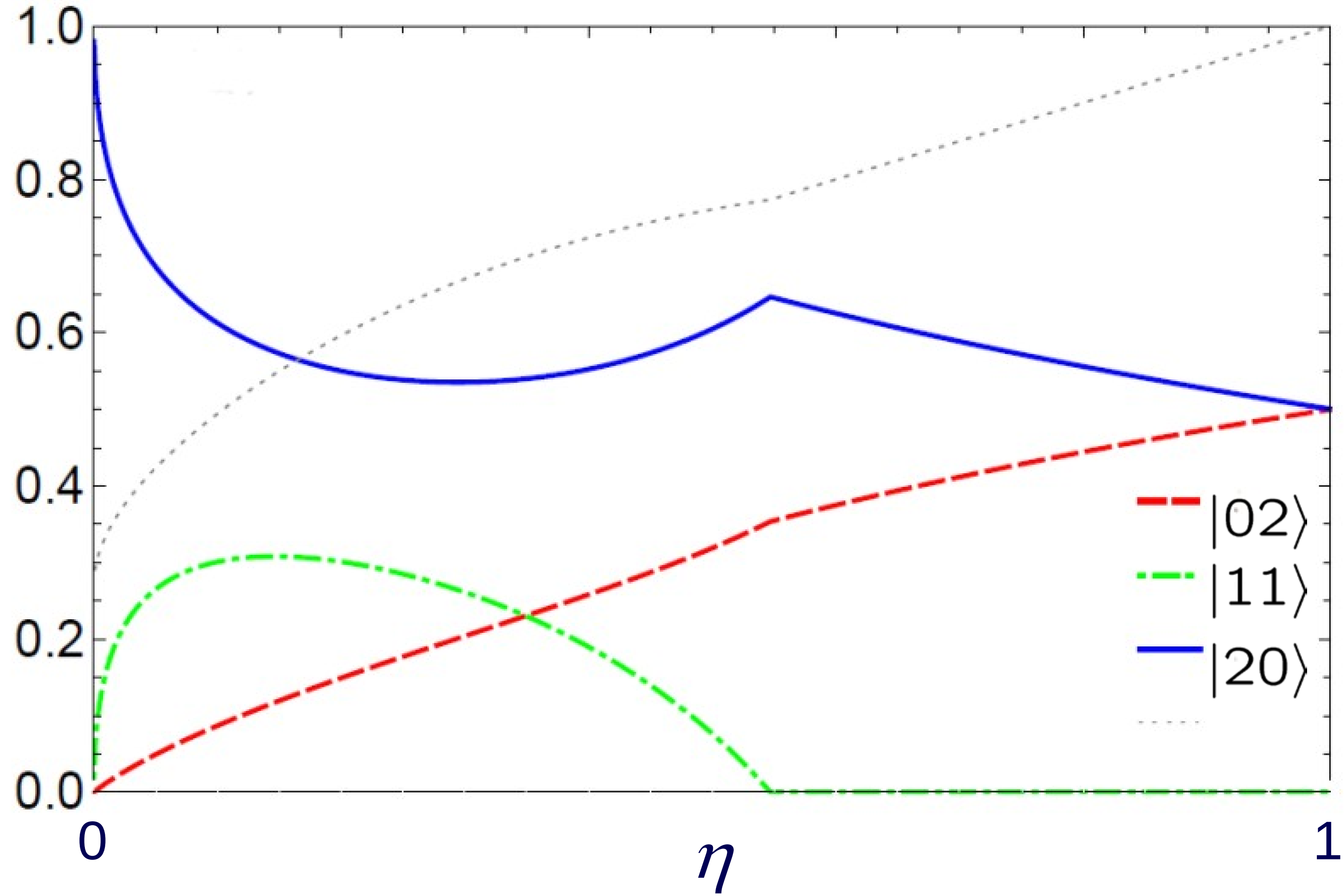
Weights



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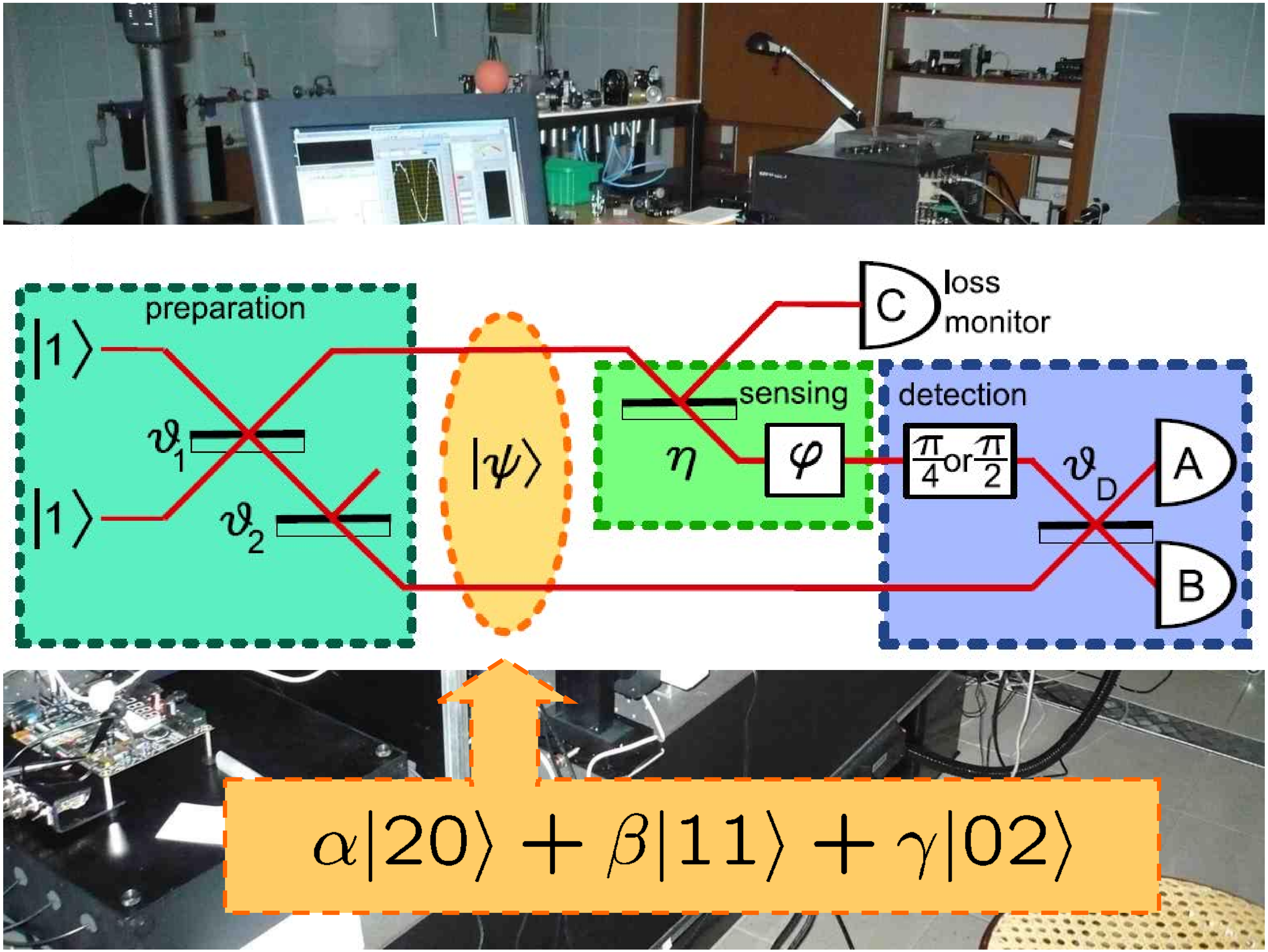
Experimental scheme



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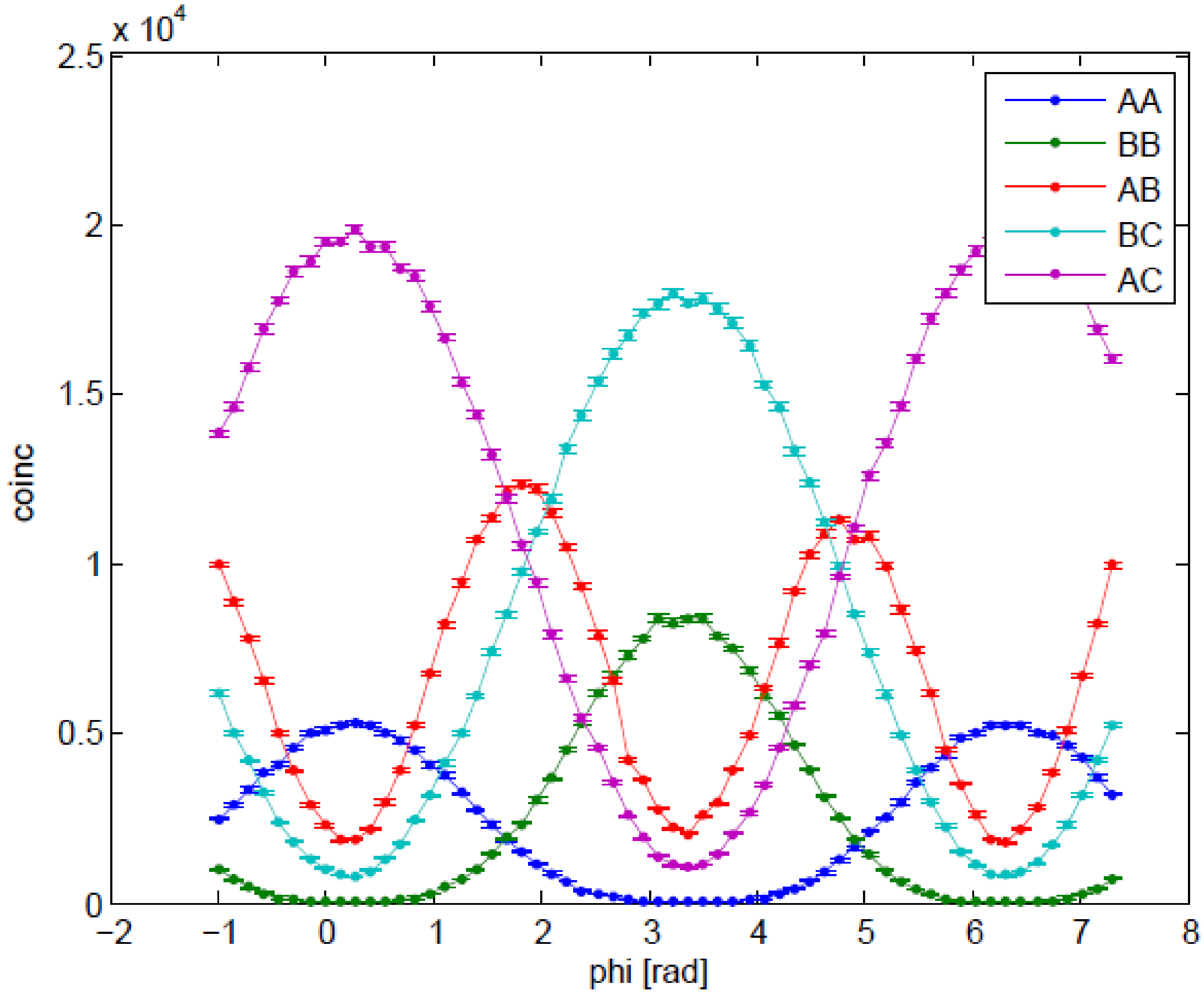
Interference pattern



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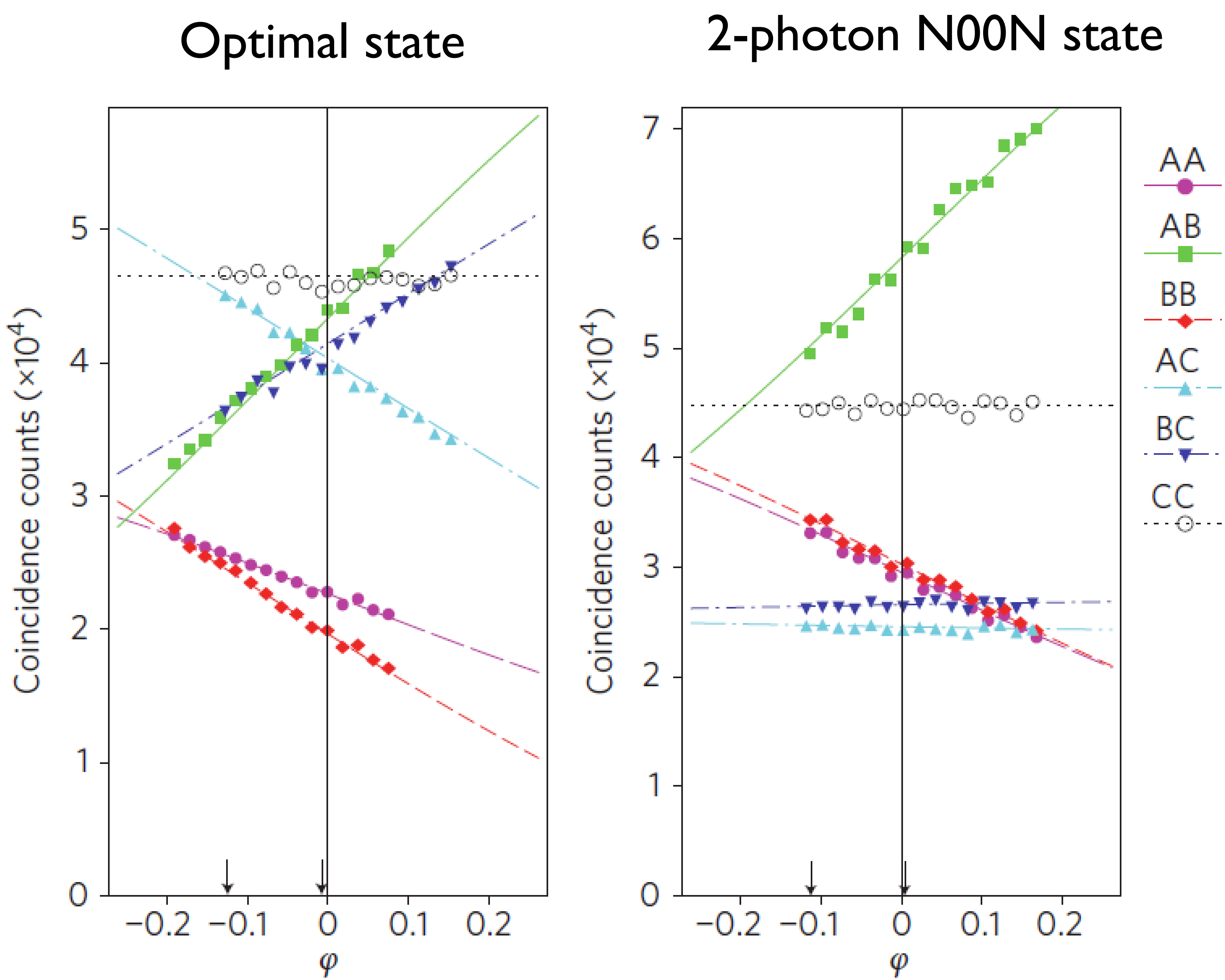
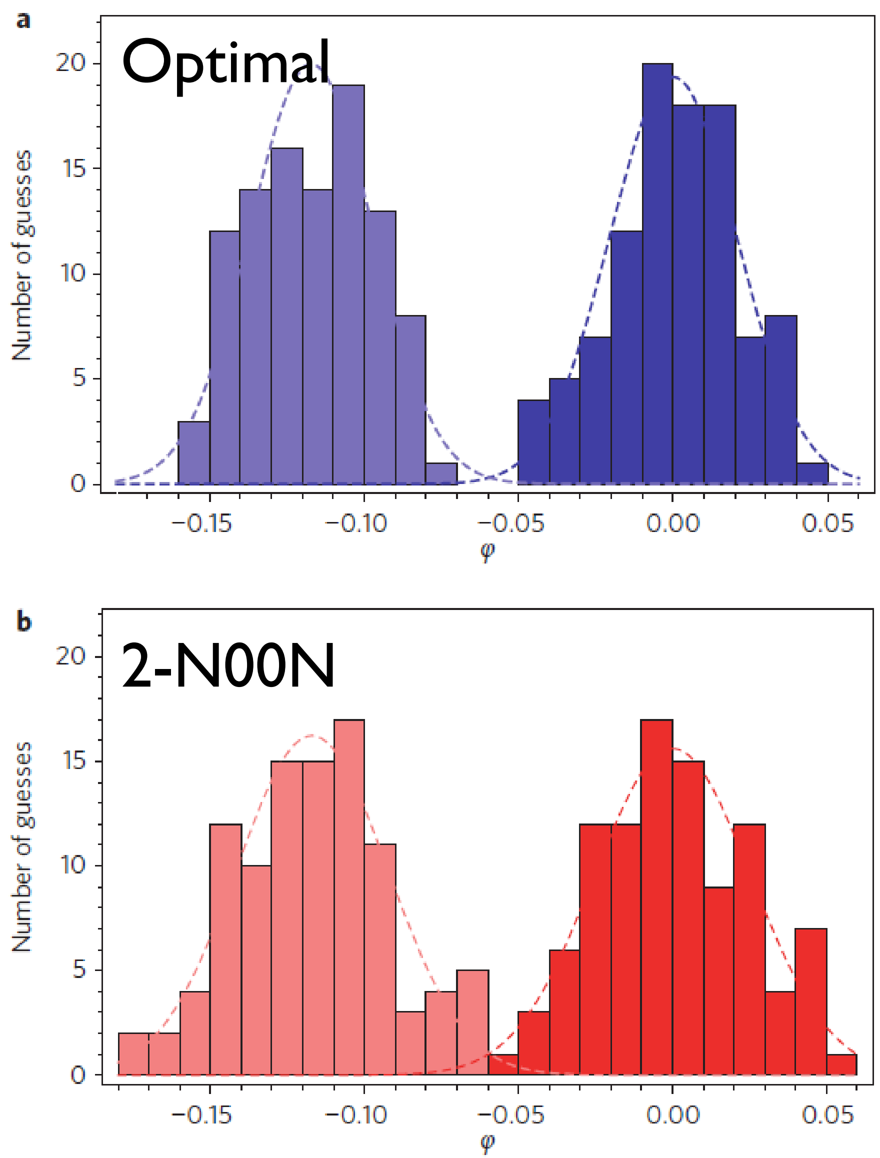
Interference fringes



Phase estimate distribution



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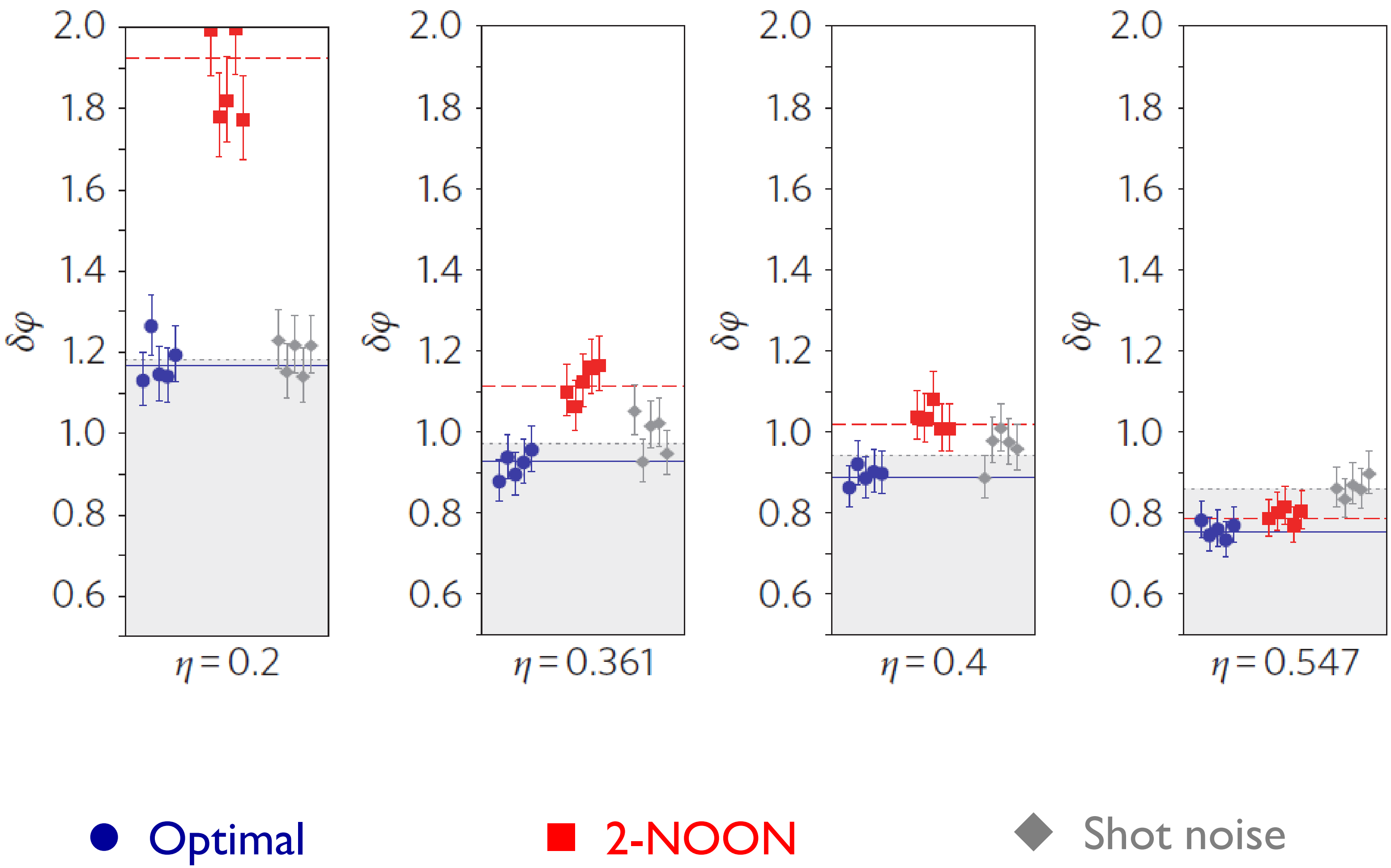
Phase estimate uncertainty



M. Kacprowicz et al.,
Nature Photon. **4**, 357 (2010)

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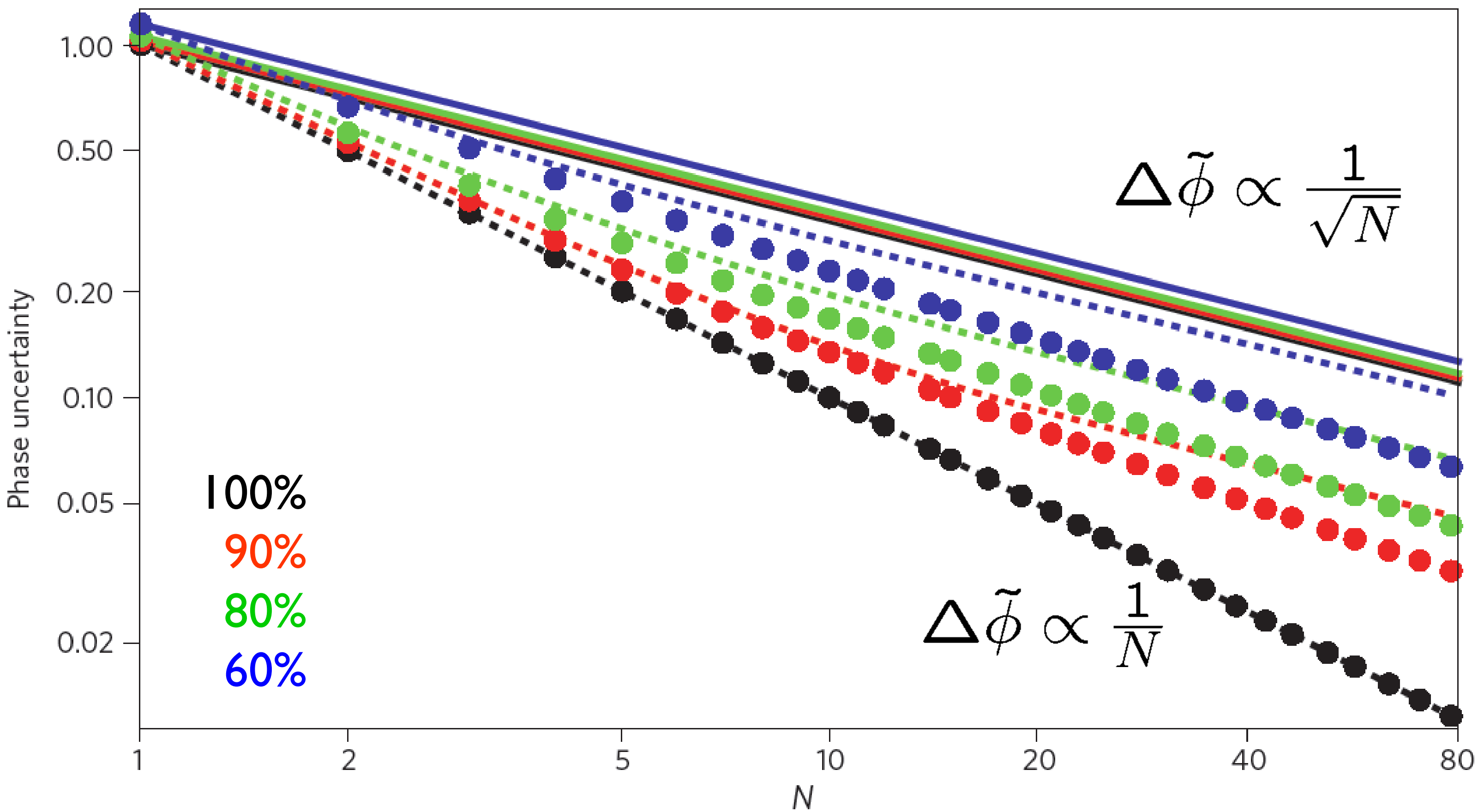
Scaling



shot noise — quantum — multipass

K. Banaszek, R. Demkowicz-Dobrzański, I. A. Walmsley, Nature Photon. **3**, 673 (2009)

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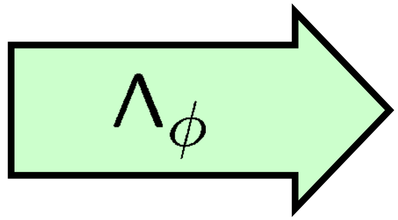
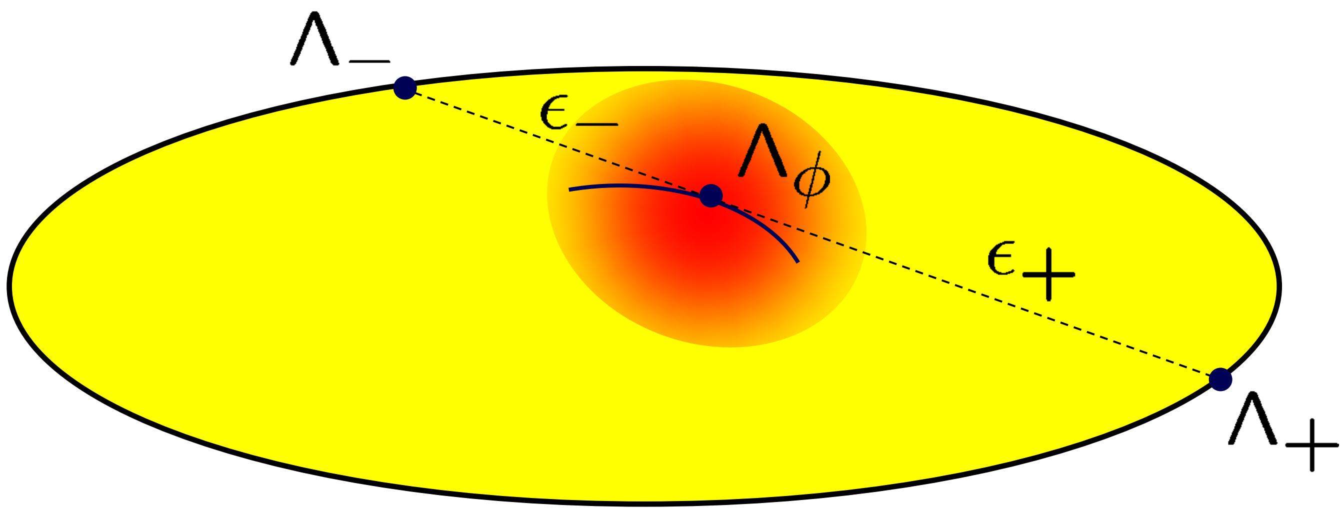
General picture

Actual value ϕ



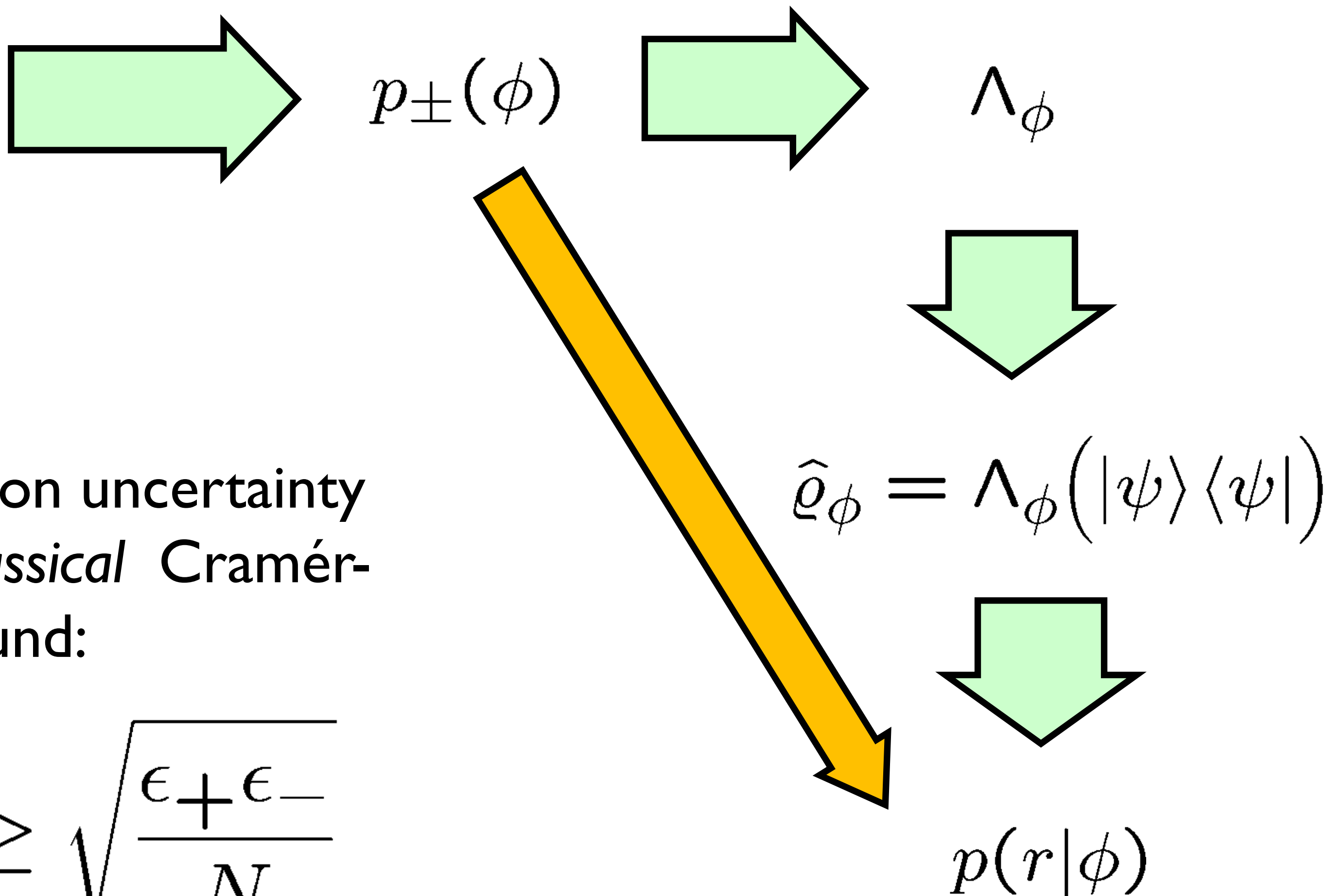
$$\hat{\varrho}_\phi = \Lambda_\phi\left(|\psi\rangle\langle\psi|\right)$$



$$\Lambda_\phi \approx p_+(\phi)\Lambda_+ + p_-(\phi)\Lambda_-$$

Classical simulation

Actual value ϕ



Estimation uncertainty from *classical* Cramér-Rao bound:

$$\Delta\tilde{\phi} \geq \sqrt{\frac{\epsilon_{+}\epsilon_{-}}{N}}$$

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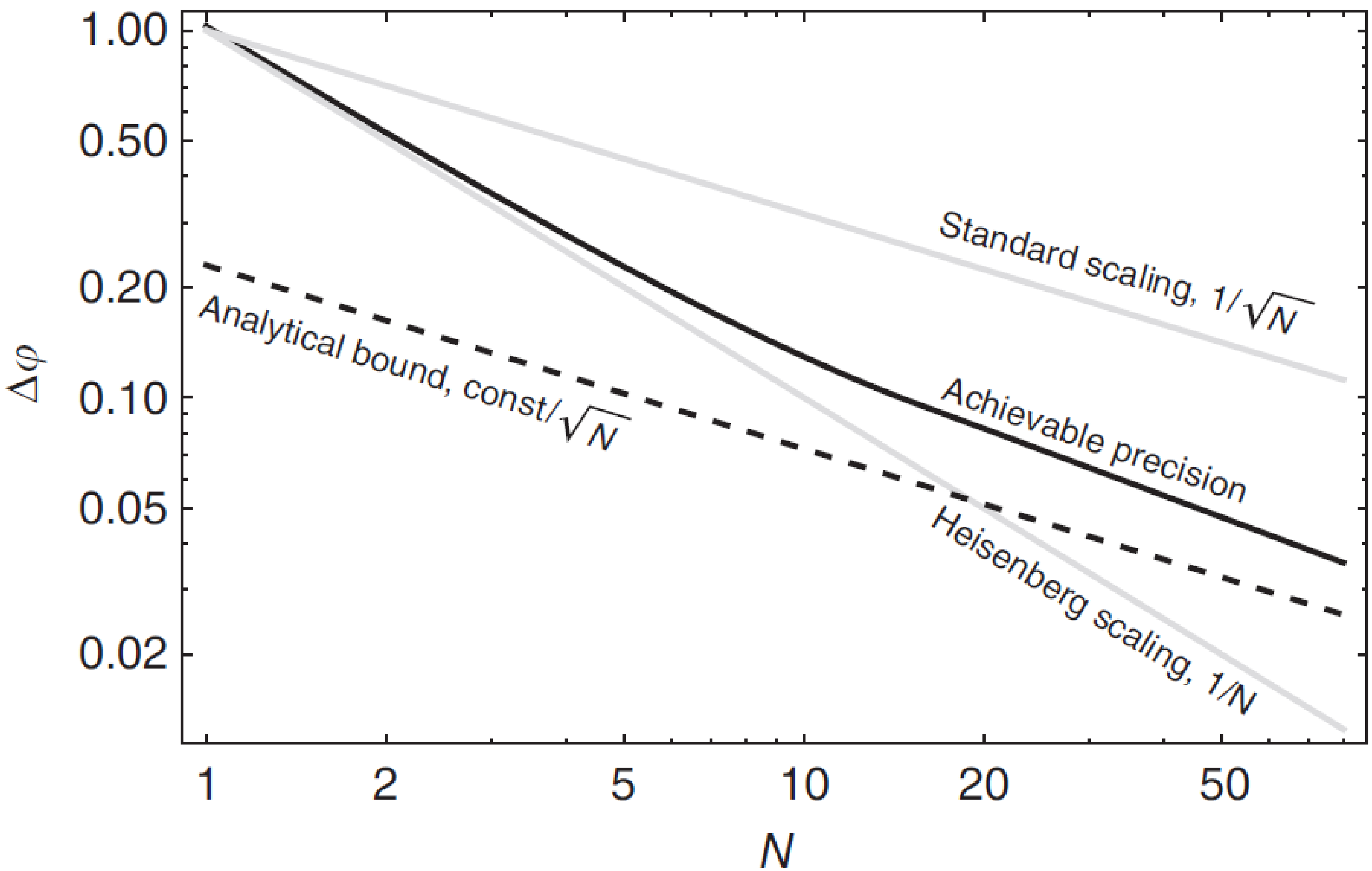
Estimation precision



R. Demkowicz-Dobrzański, J. Kołodyński, and M. Guţă, Nature Comm. **3**, 1063 (2012)

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Asymptotic bounds

Table 1 | Precision bounds of the most relevant models in quantum-enhanced metrology.

Channel considered	Classical simulation	Channel extension
Depolarisation	$\sqrt{(1-\eta)(1+3\eta)/4\eta^2}$	$\sqrt{(1-\eta)(1+2\eta)/2\eta^2}$
Dephasing	$\sqrt{1-\eta^2/\eta}$	$\sqrt{1-\eta^2/\eta}$
Spontaneous emission	NA	$(1/2)\sqrt{1-\eta/\eta}$
Lossy interferometer	NA	$\sqrt{1-\eta/\eta}$

NA, not available.
The bounds are derived using the two methods discussed in the paper. All the bounds are of the form $\Delta\varphi_N \geq (\mathrm{const}/\sqrt{N})$, where constant factors are given in the table. Classical simulation method does not provide bounds for spontaneous emission and lossy interferometer, as these channels are φ-extremal. For the dephasing model, it surprisingly yields an equally tight bound as the more powerful channel extension method.

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R. Demkowicz-Dobrzański, J. Kołodyński, and M. Guţă, Nature Comm. **3**, 1063 (2012)

Undefined photon number

When no external phase reference is used:

$$\hat{\varrho} = \bigoplus_{N=0}^{\infty} p_n \hat{\varrho}_N$$

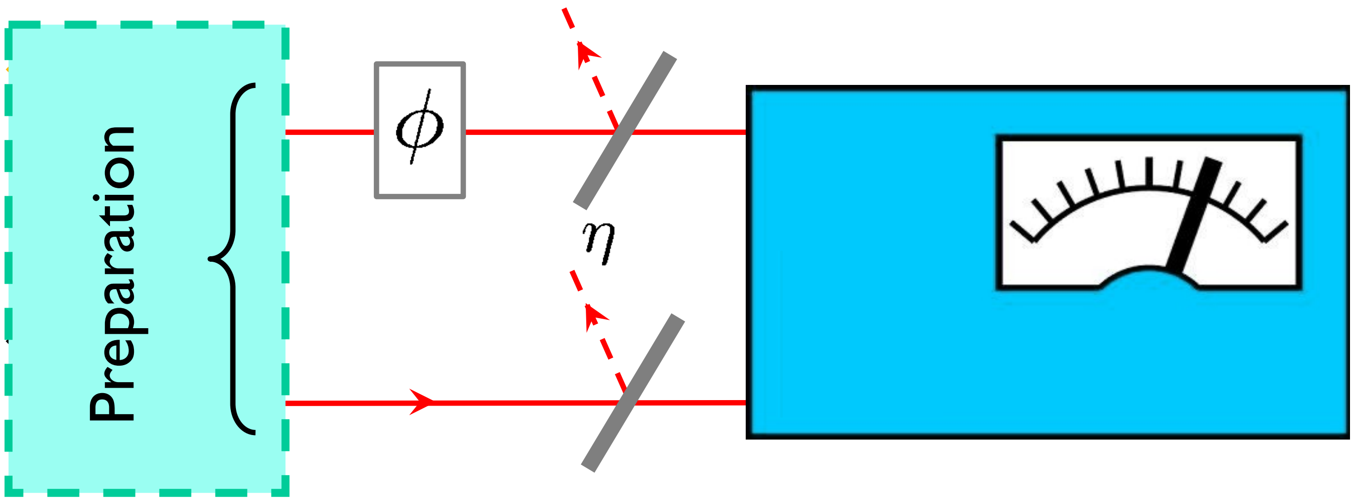
Convexity of Fisher information:

$$\mathsf{F}(\hat{\varrho}) \leq \sum_{N=0}^{\infty} p_n \mathsf{F}(\hat{\varrho}_N)$$

Using the bound for a fixed photon number:

$$\leq \sum_{N=0}^{\infty} p_n \cdot \mathsf{const} \cdot N = \mathsf{const} \cdot \langle N \rangle$$

Two-arm losses



For a quantum state with $\langle N \rangle$ average photon number

Shot noise limit

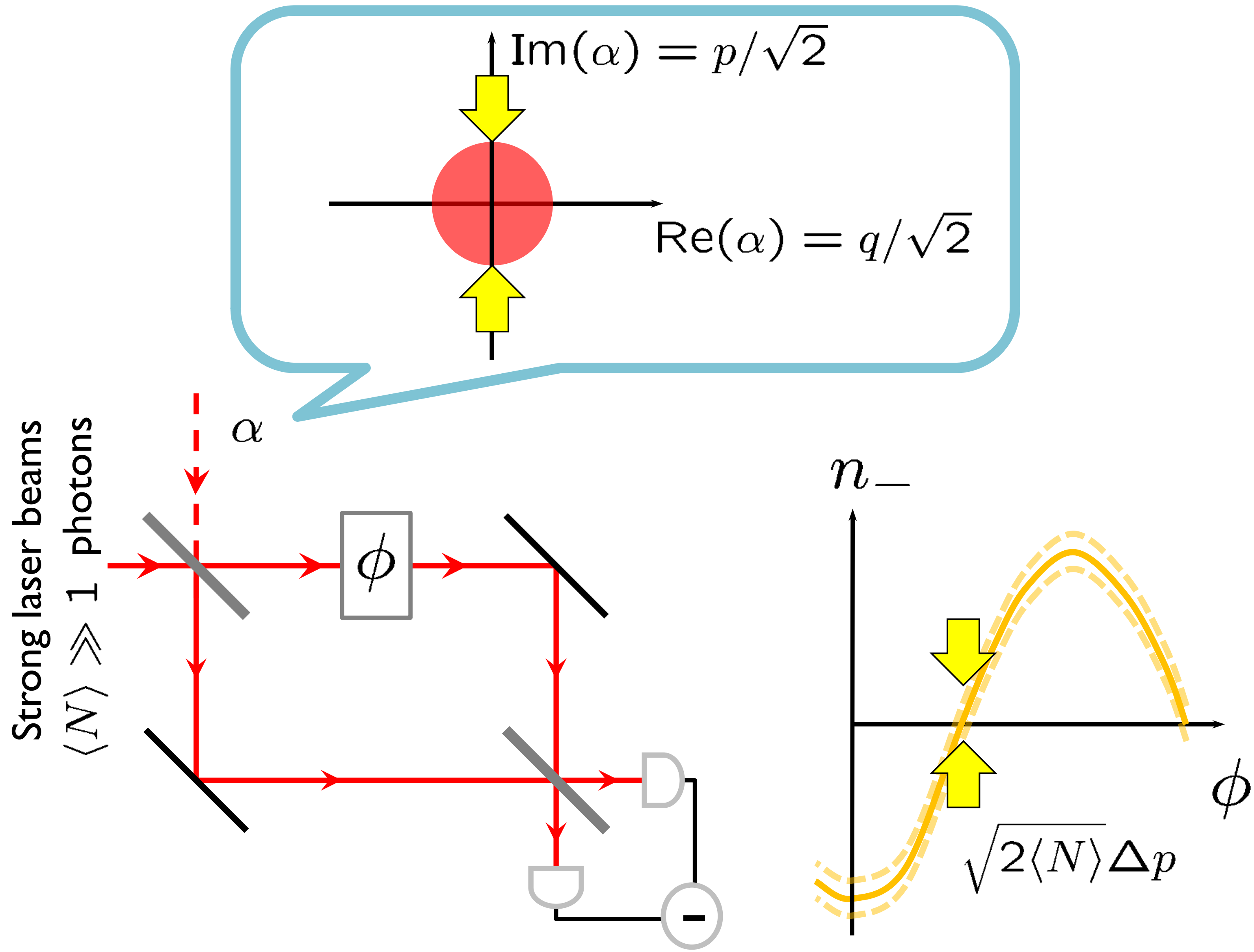
$$\Delta\tilde{\phi} \geq \frac{1}{\sqrt{\eta\langle N\rangle}}$$

Ultimate quantum limit

$$\Delta\tilde{\phi} \geq \sqrt{\frac{1-\eta}{\eta\langle N\rangle}}$$

*Assuming no external phase reference is available

Shot noise revisited



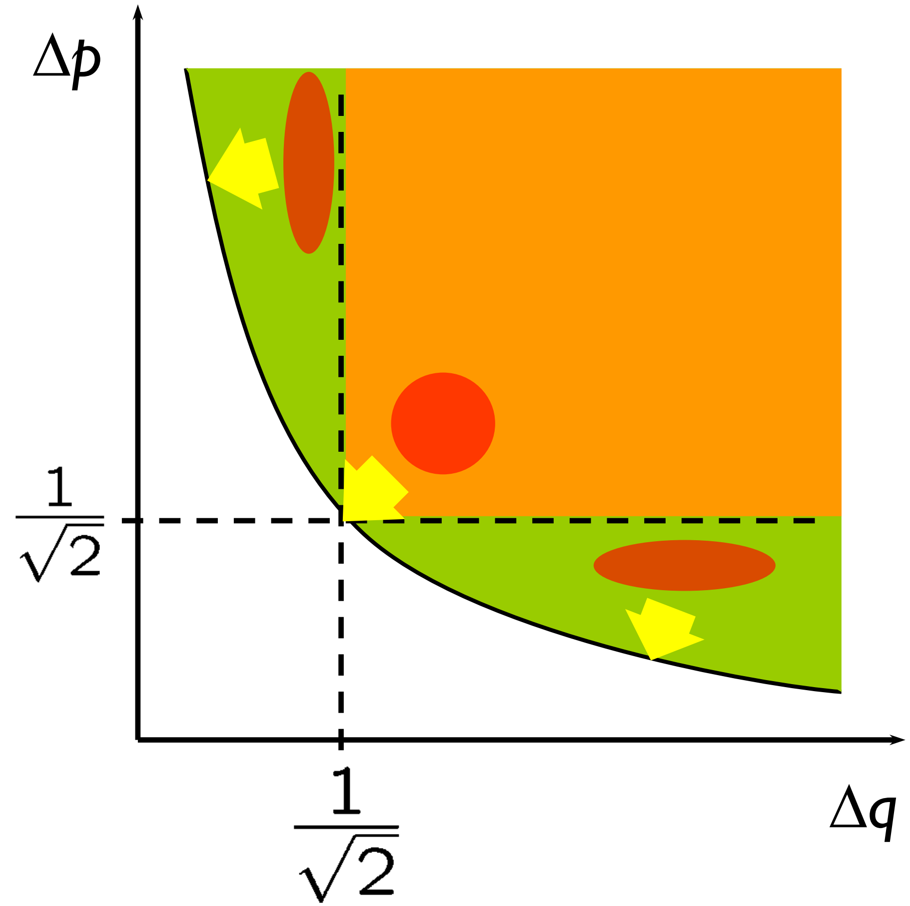
Quadrature uncertainties

Conjugate variables:

$$[\hat{q}, \hat{p}] = i$$

Uncertainty relation:

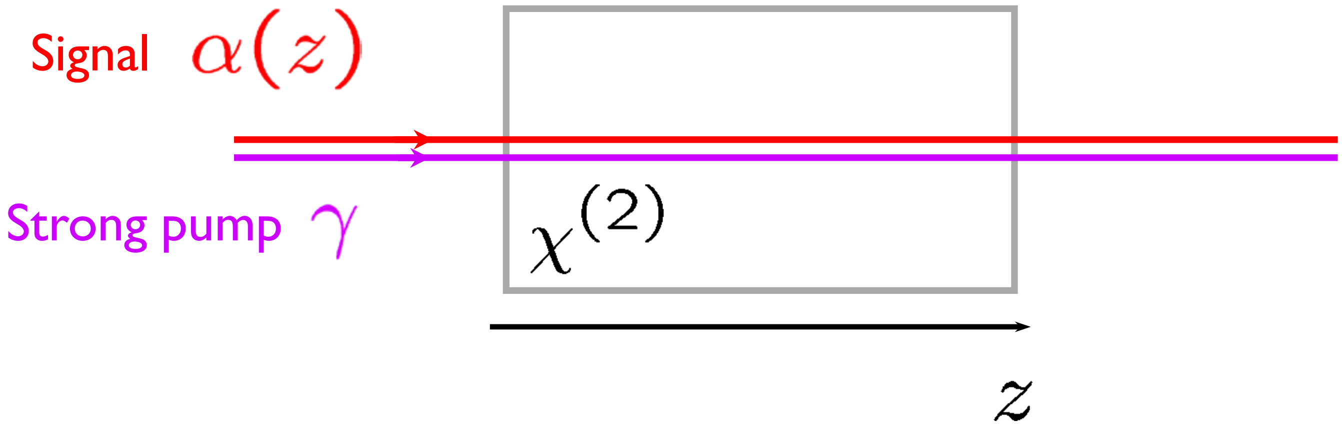
● (green) $$\Delta q \cdot \Delta p \geq \frac{1}{2}$$



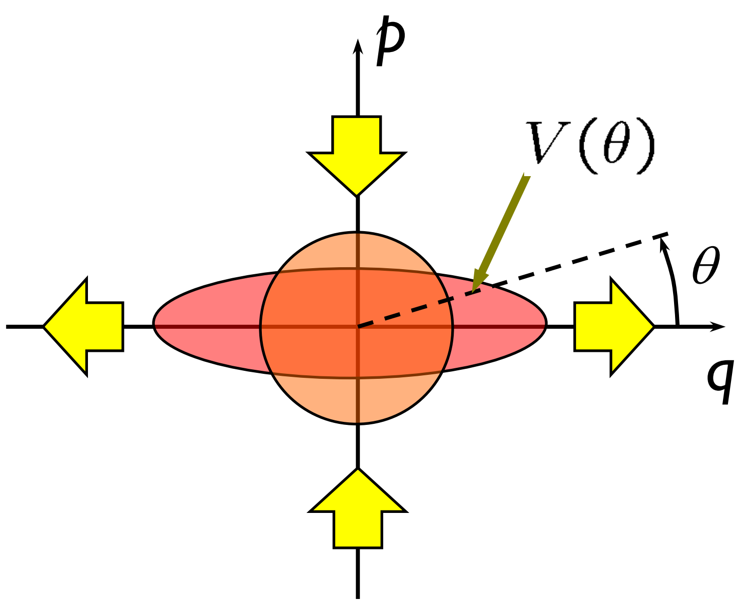
For classical light (laser sources + noise)

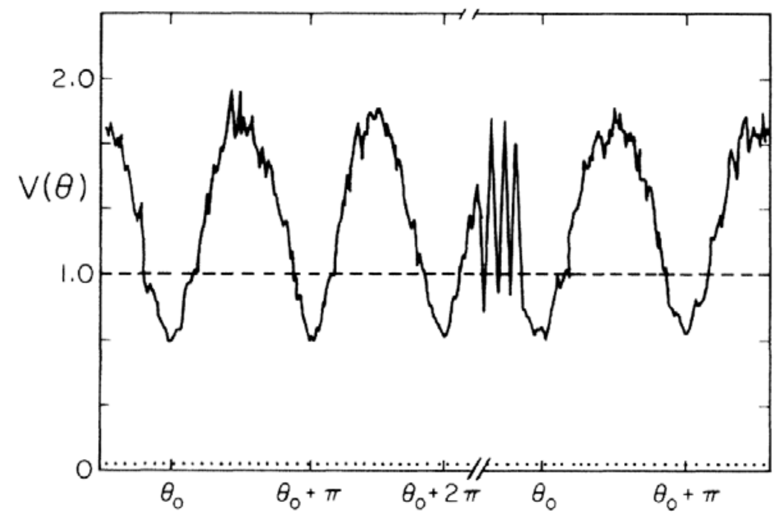
● (orange) $$\Delta q \geq \frac{1}{\sqrt{2}} \text{ AND } \Delta p \geq \frac{1}{\sqrt{2}}$$

Squeezed states



Amplitude evolution:
$$\frac{\mathrm{d}\alpha}{\mathrm{d}z} = \kappa\gamma\alpha^*(z)$$





L.-A. Wu *et al.*, Phys. Rev. Lett. **57**, 2520 (1986)

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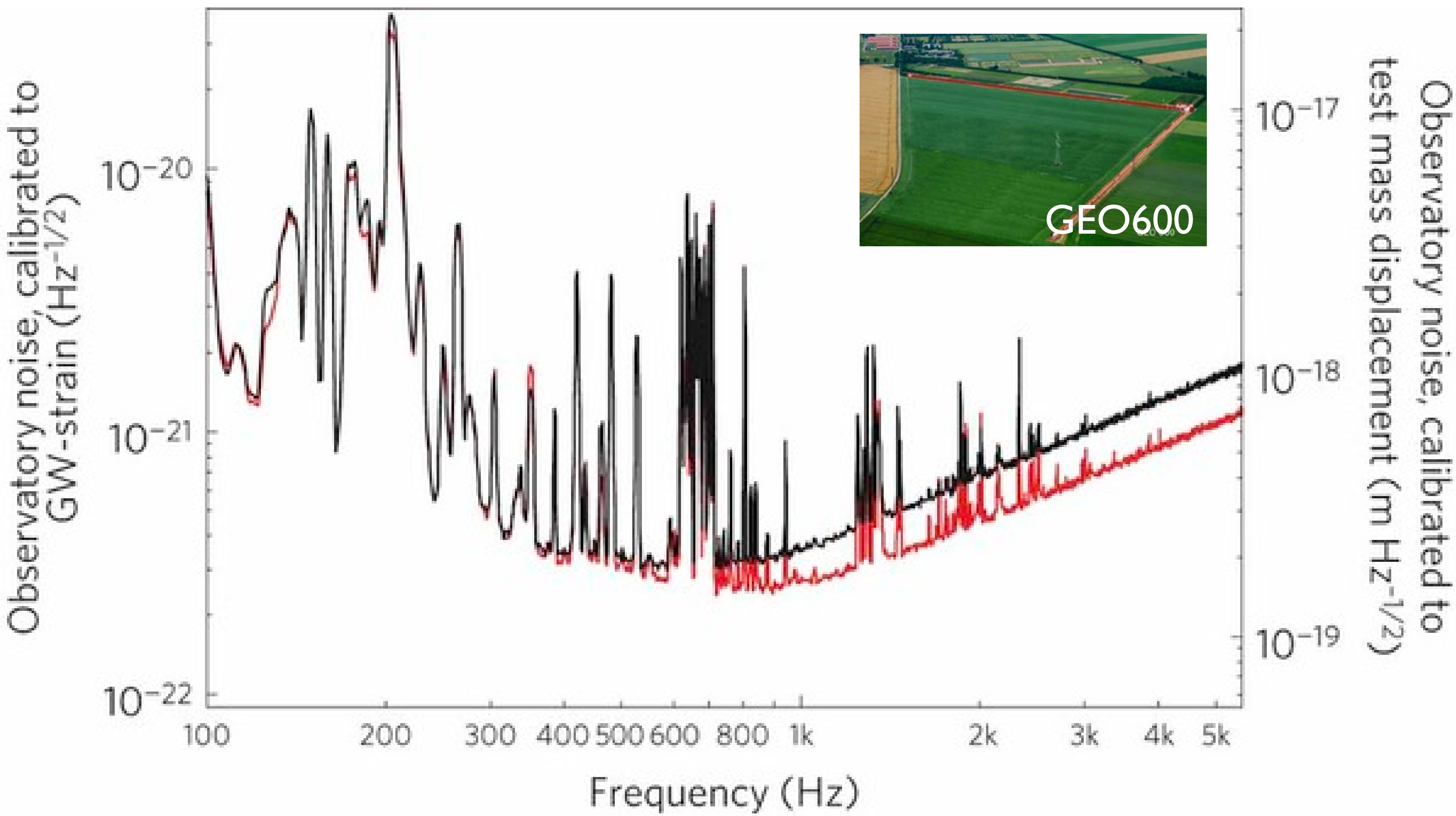
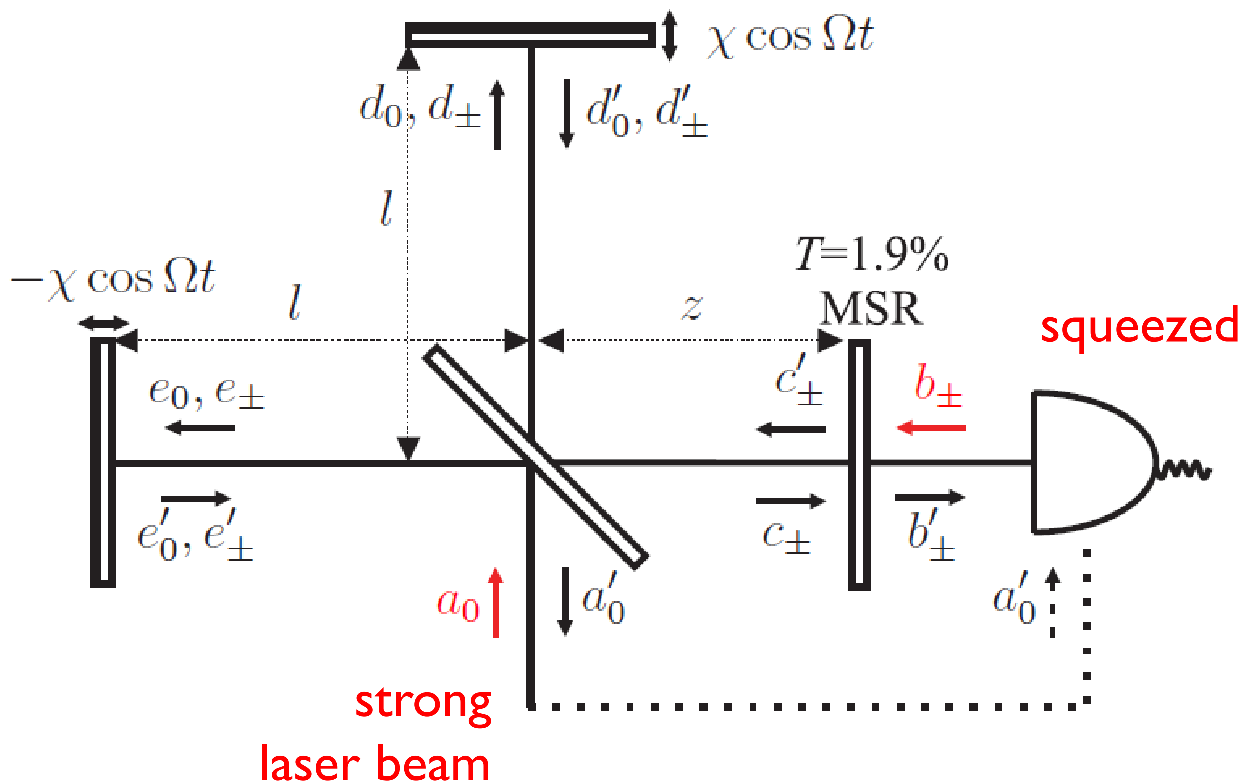
Gravitational wave detection



$$\frac{\Delta\tilde{\phi}_{\text{squeezed}}}{\Delta\tilde{\phi}_{\text{shot noise}}} \approx 0.66$$

J. Abadie *et al.* (The LIGO Scientific Collaboration), Nature Phys. **7**, 962 (2011)

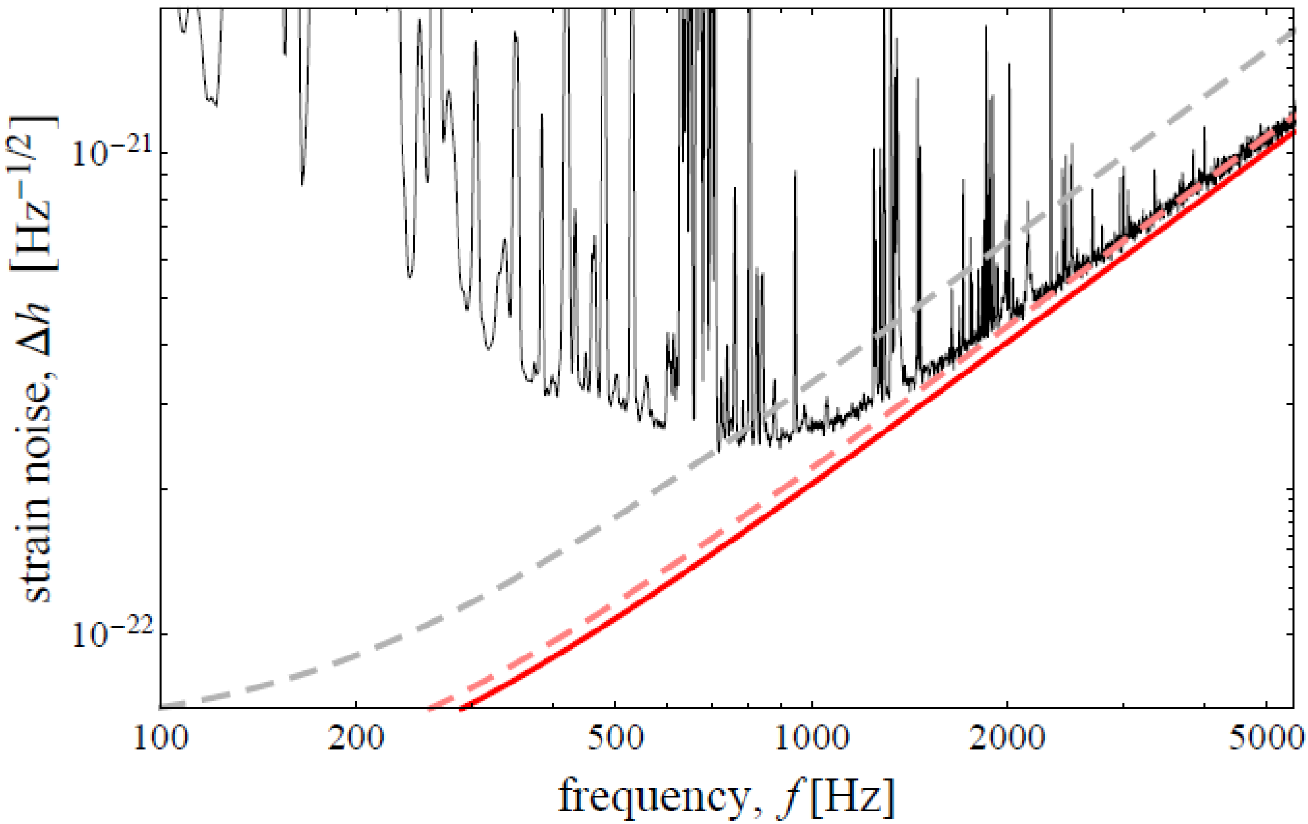
Theoretical model



When most power comes from the laser beam

$$\Delta\tilde{\phi} \approx \sqrt{\frac{1-\eta+2\eta(\Delta p)^2}{\eta\langle N\rangle}}$$

Noise analysis



Shot noise limit

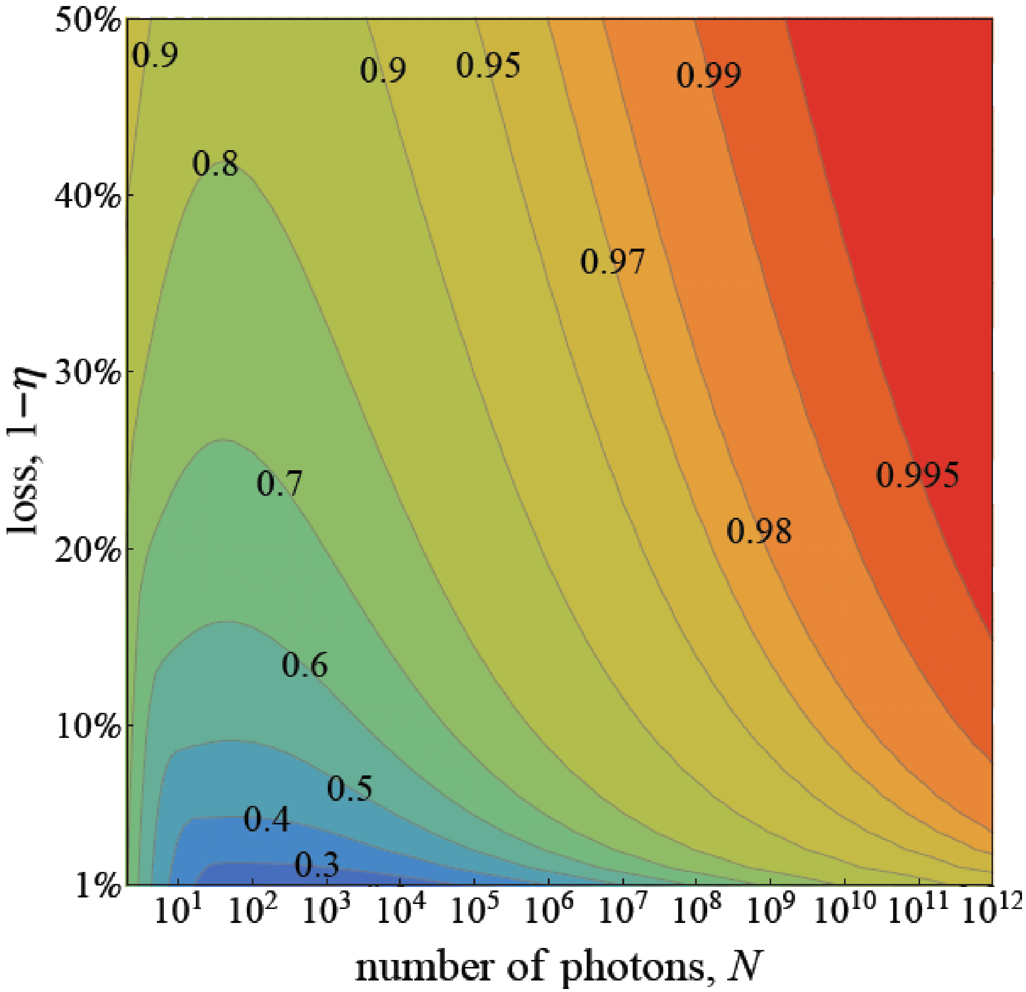
10dB squeezing (implemented)

16dB squeezing and ultimate bound

Overall interferometer transmission $\eta \approx 62\%$

R. Demkowicz-Dobrzański, K. Banaszek, and R. Schnabel, Phys. Rev. A **88**, 041802(R) (2013)

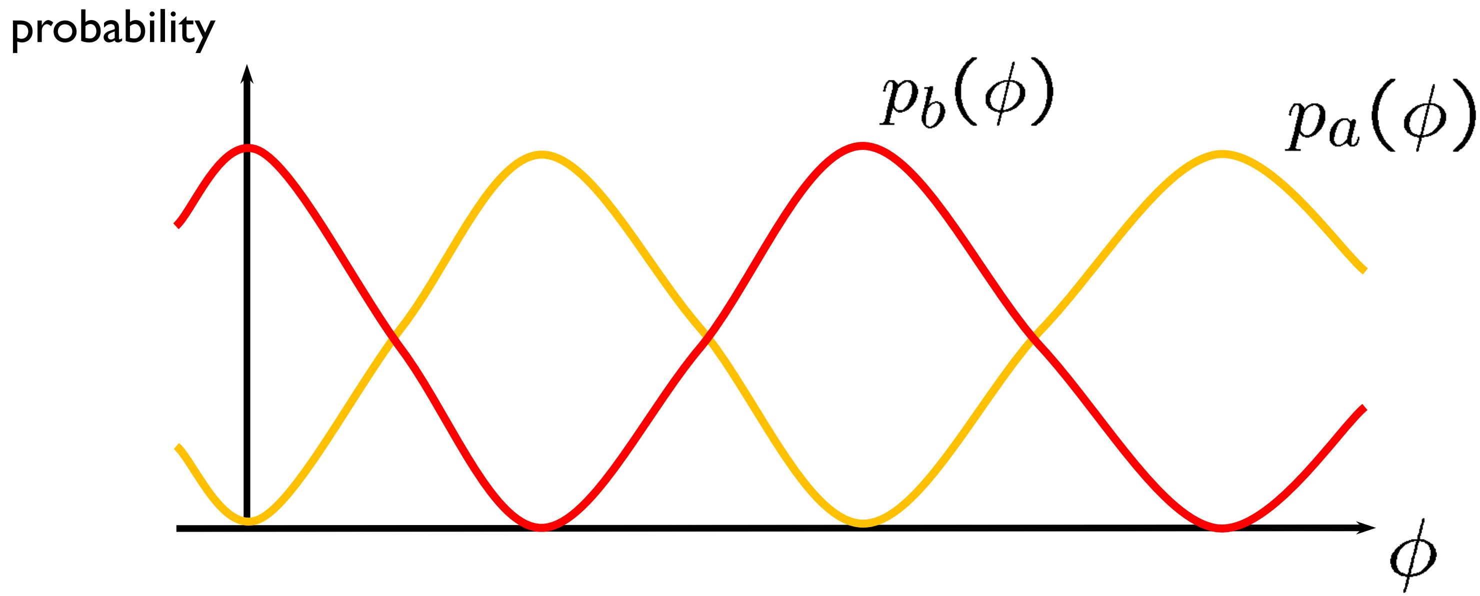
Optimality of squeezed states



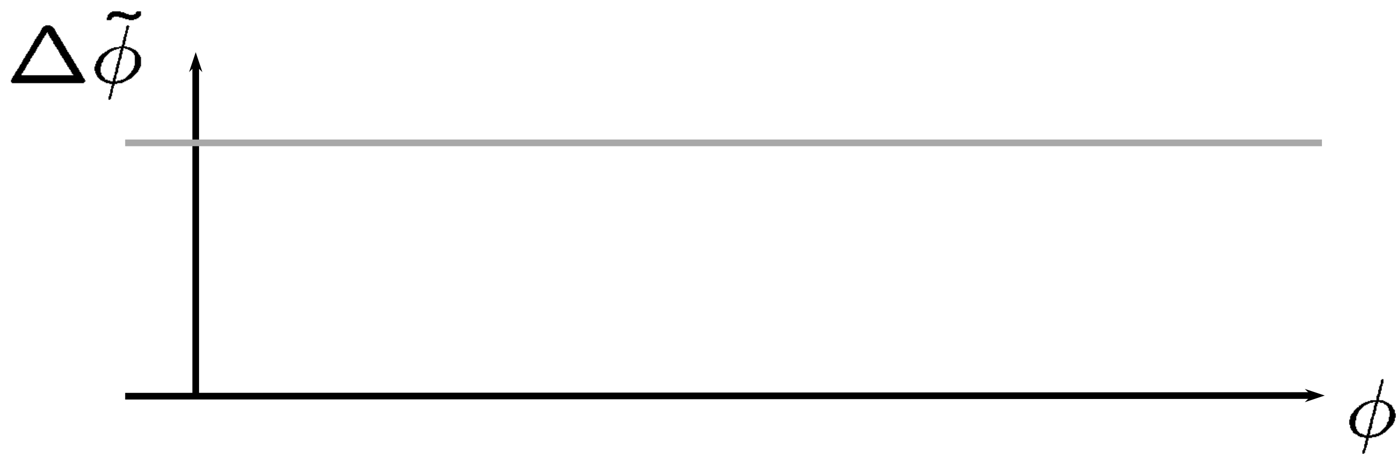
$$\frac{\Delta\tilde{\phi}_{\text{optimal}}}{\Delta\tilde{\phi}_{\text{squeezed}}}$$

R. Demkowicz-Dobrzański,
K. Banaszek, and R. Schnabel,
Phys. Rev. A **88**, 041802(R) (2013)

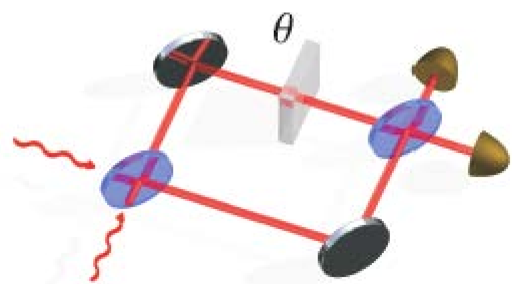
Operating point

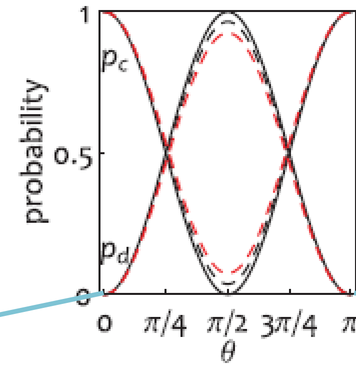


Uncertainty of phase estimated from (n_a, n_b)



Residual distinguishability

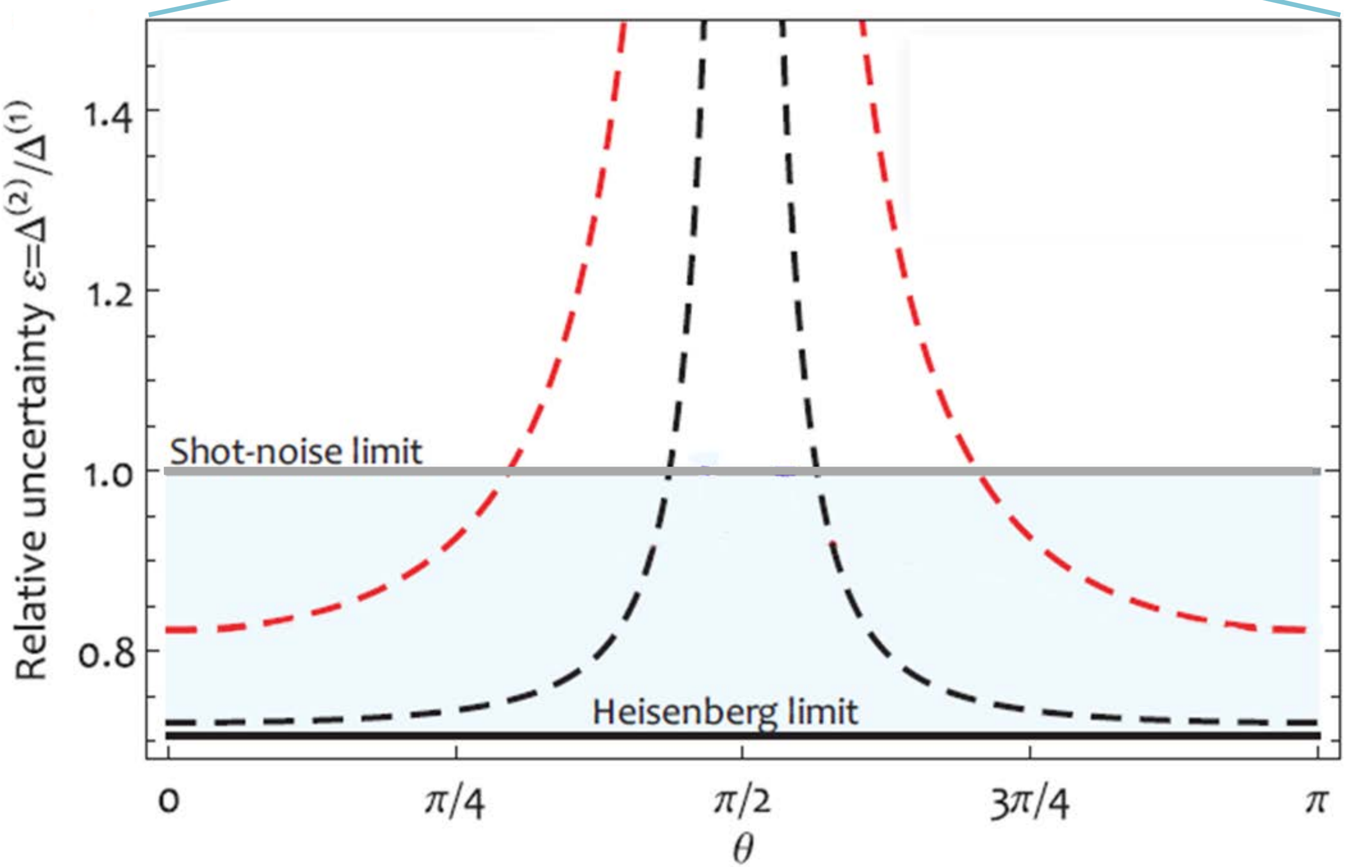




Fraction of indistinguishable photon pairs V:

- - - - - - - 93%

- - - - - - - 85%



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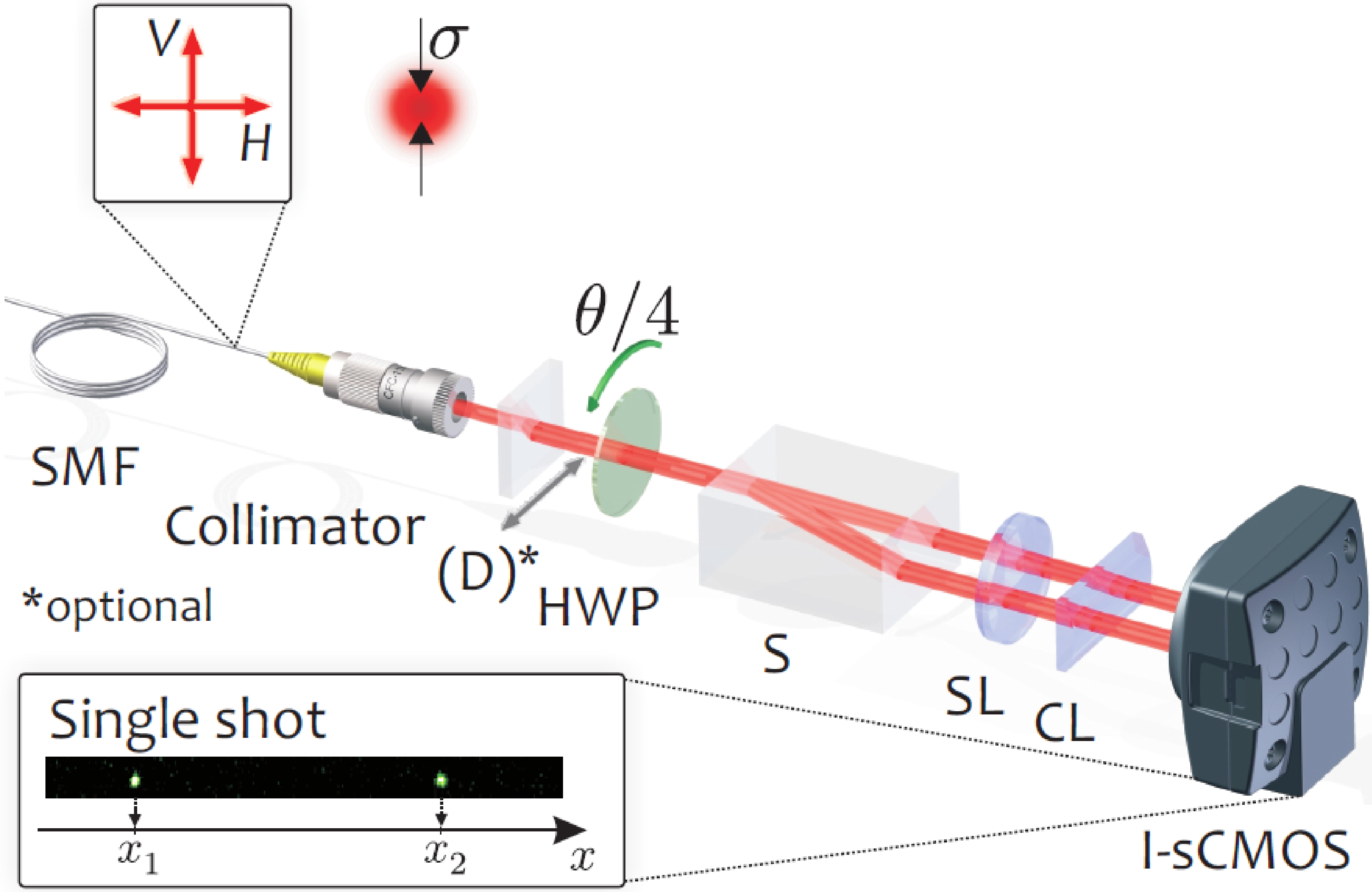
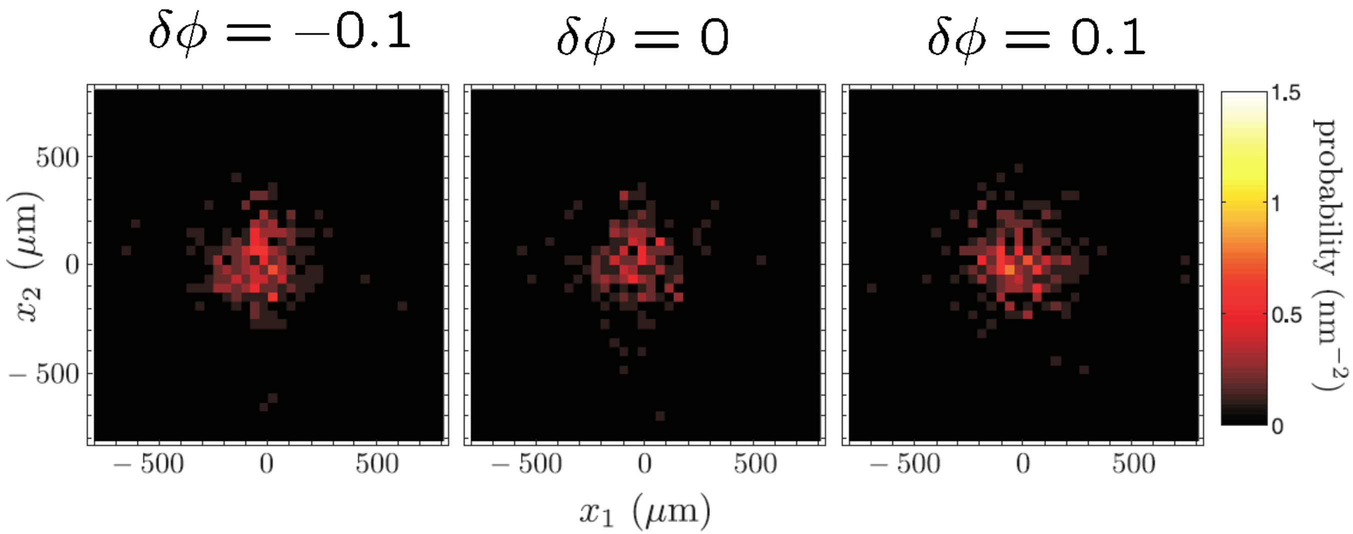
Imaging experiment

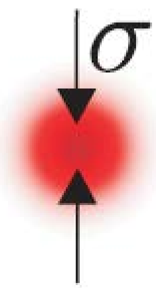


M. Jachura, R. Chrapkiewicz *et al.*, arXiv:1504.05435

Joint output position distribution

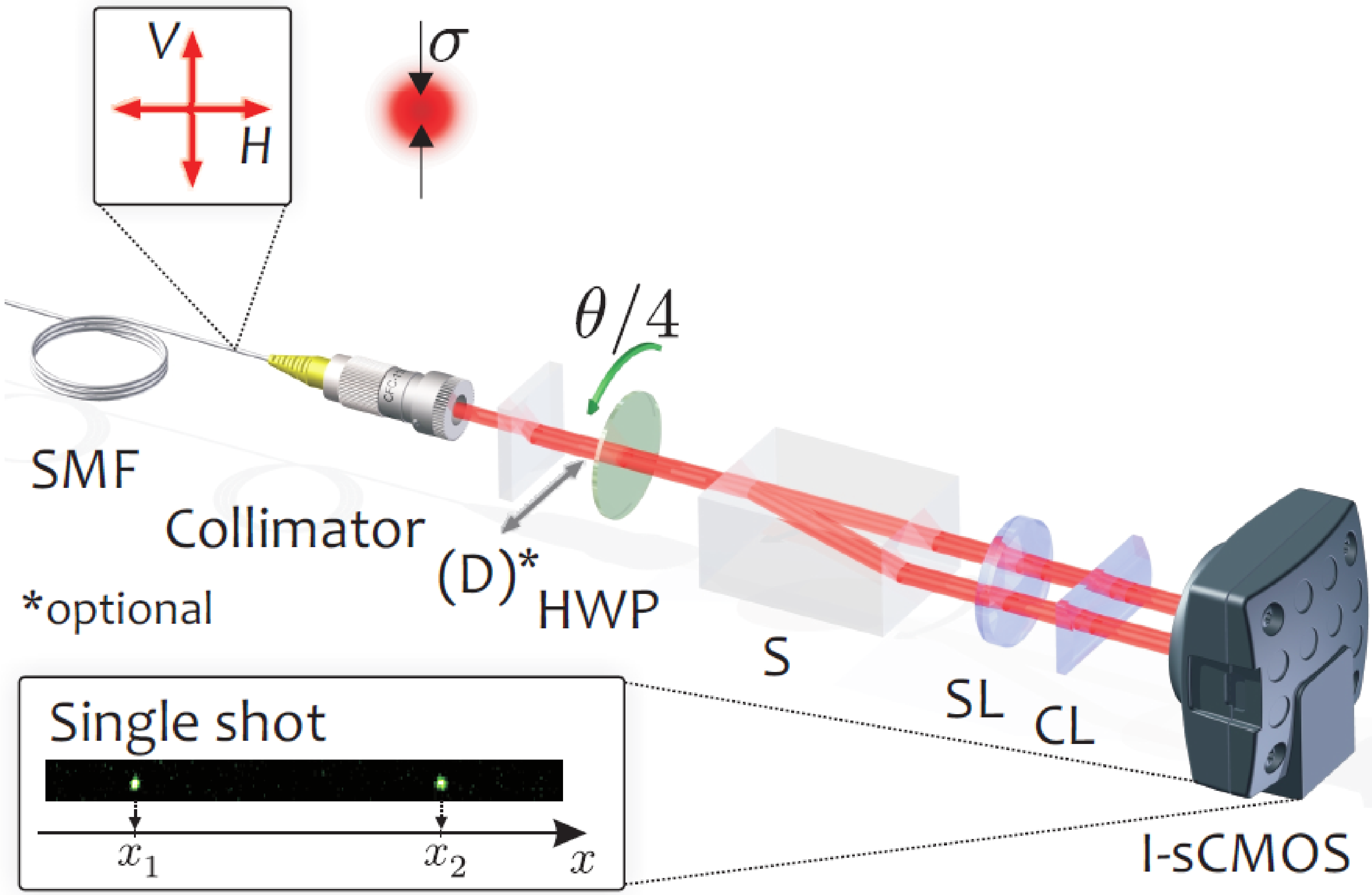


Input mode:



M. Jachura, R. Chrapkiewicz *et al.*, arXiv:1504.05435

Imaging experiment



M. Jachura, R. Chrapkiewicz *et al.*, arXiv:1504.05435

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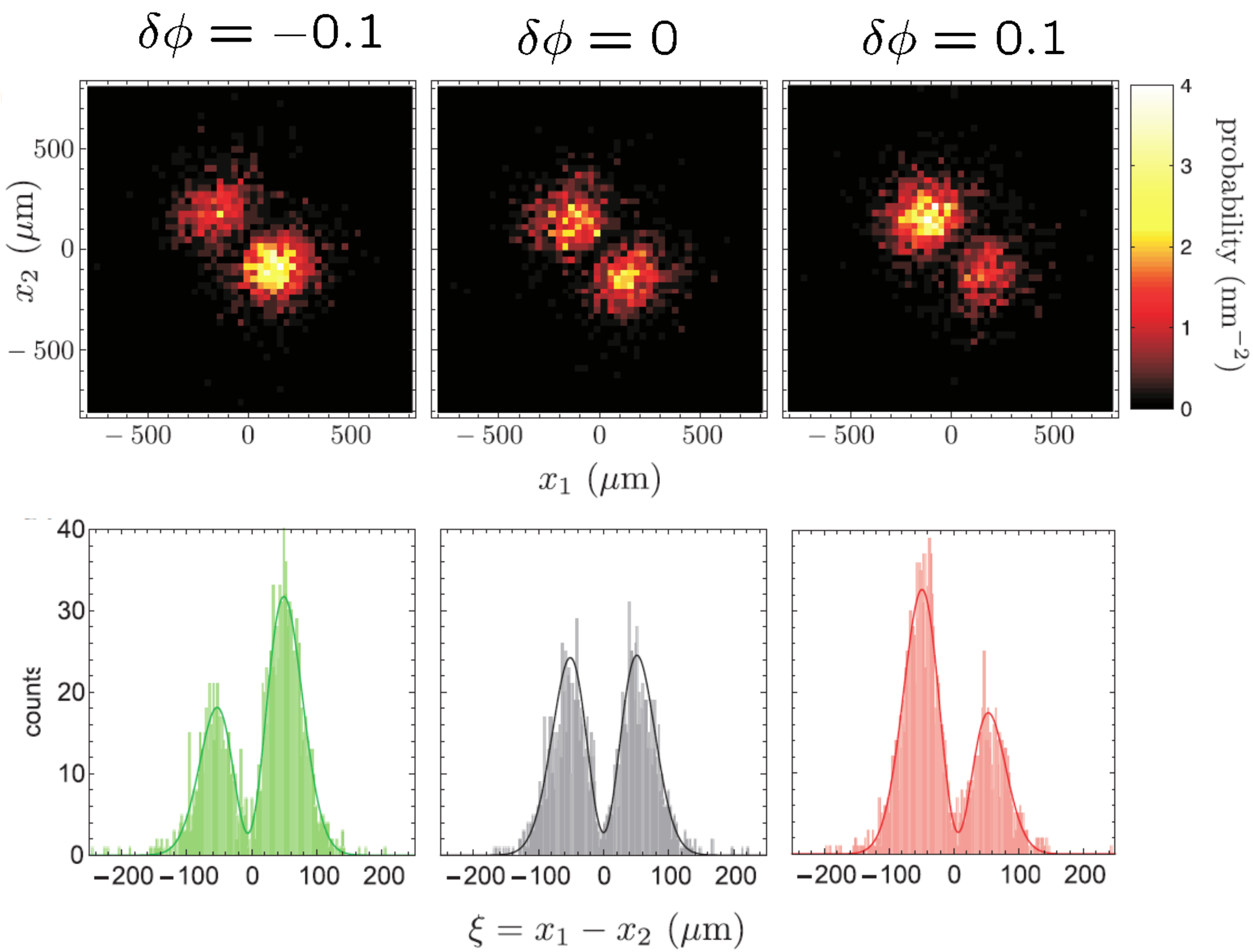
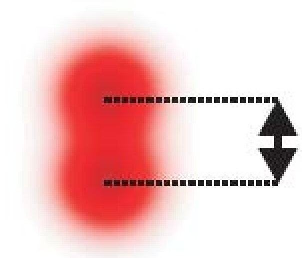


Input mode:



$$p_c(\phi) \rightarrow p_c(\xi|\phi)$$

M. Jachura, R. Chrapkiewicz *et al.*, arXiv:1504.05435

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Phase shift estimation

Fisher information

$$F(\phi) = F_c(\phi) + \frac{1}{p_d(\phi)} \left(\frac{\partial p_d}{\partial \phi} \right)^2$$

Contribution from spatial distribution:

$$F_c(\phi) = \int \mathrm{d}\xi \frac{1}{p_c(\xi|\phi)} \left(\frac{\partial}{\partial \phi} p_c(\xi|\phi) \right)^2$$

Local estimator:

$$\Phi[f] = \phi_0 + \frac{1}{F_c(\phi_0)} \int \mathrm{d}\xi \, \frac{f(\xi)}{p_c(\xi|\phi_0)} \left. \frac{\partial p_c(\xi|\phi)}{\partial \phi} \right|_{\phi=\phi_0}$$

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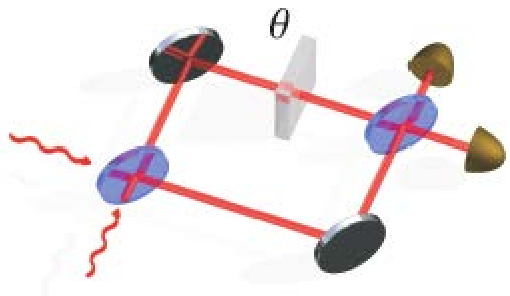
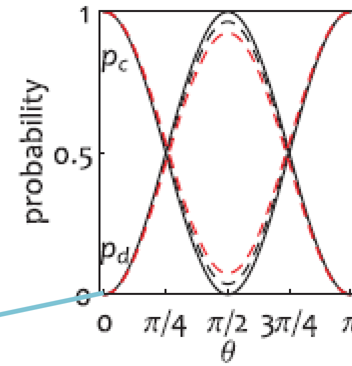
Residual distinguishability





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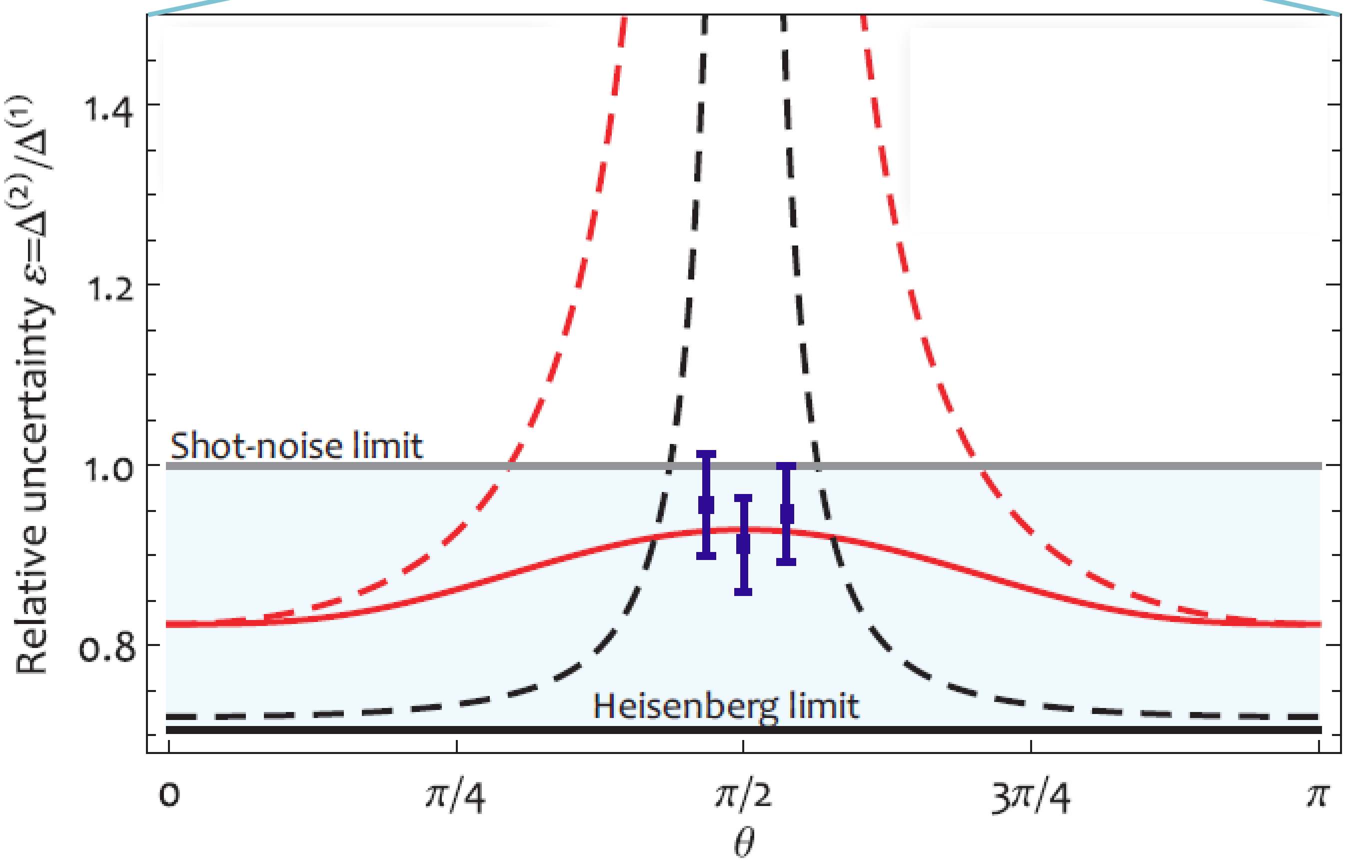

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University of Nottingham

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Universität Hamburg

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SEVENTH FRAMEWORK PROGRAMME

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