Quantum atom optics with Bose condensates
Part 1

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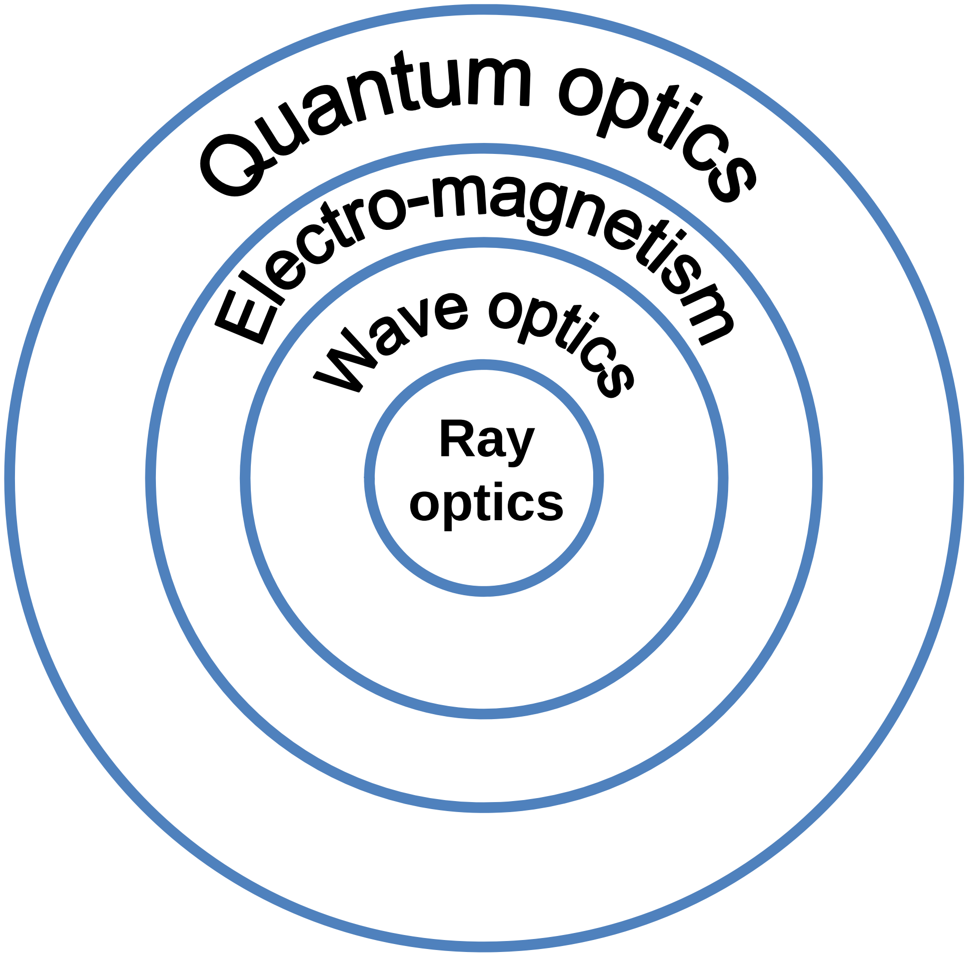
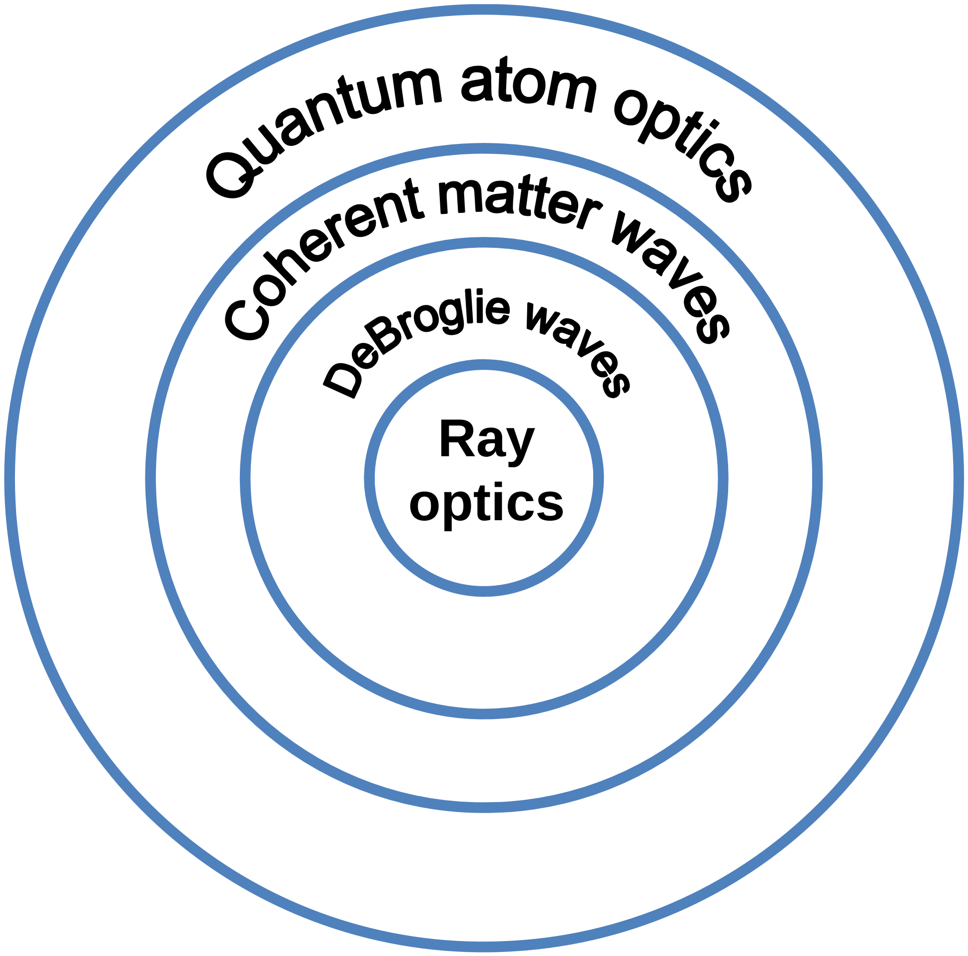
The hierarchy of optics

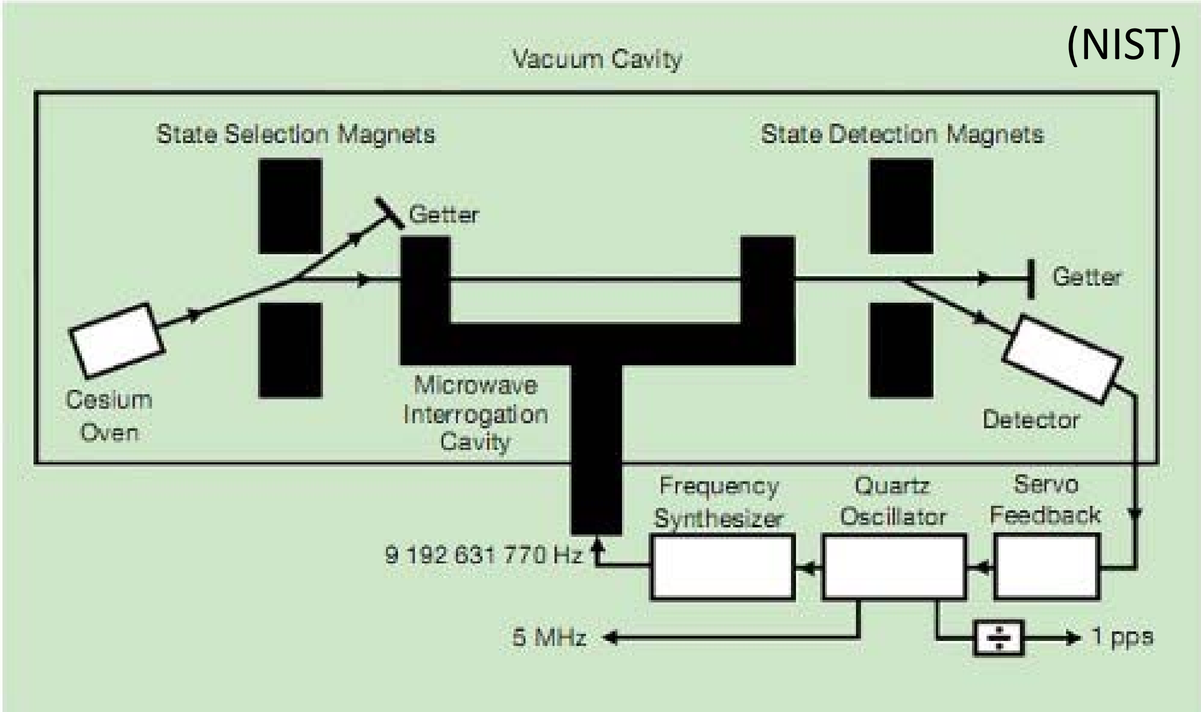


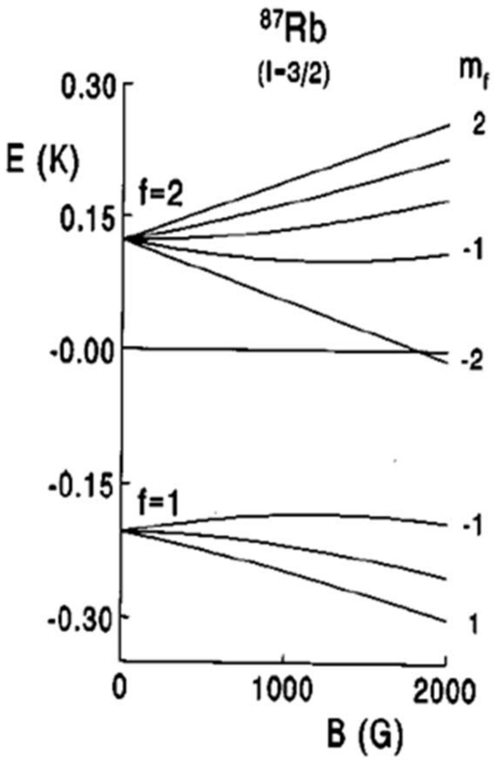
The hierarchy of atom optics



Ray optics: atomic beams

- Collimation
- Beam steering
- Focusing
- Early days:
  - Quantum state preparation
  - Quantum state measurement



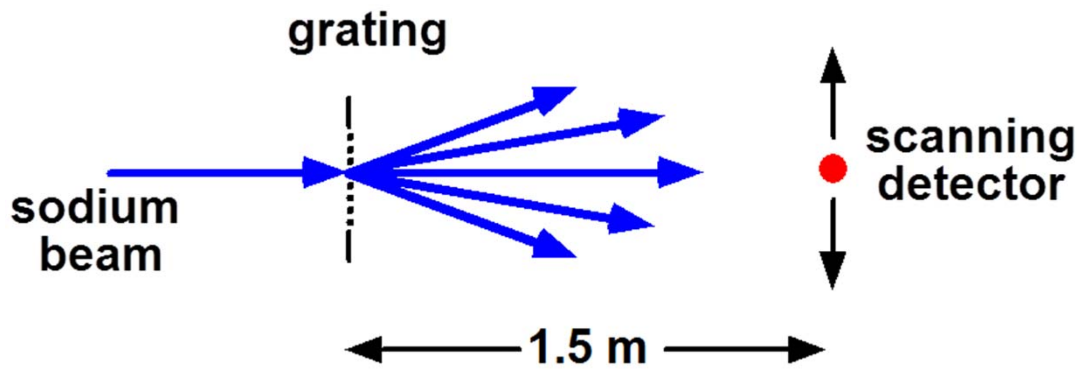


$$E(\mathbf{r}) = \vec{\mu} \cdot \vec{B}(\mathbf{r})$$

Spatial field gradiant ←→ magnetic force

Atom wave optics: deBroglie waves

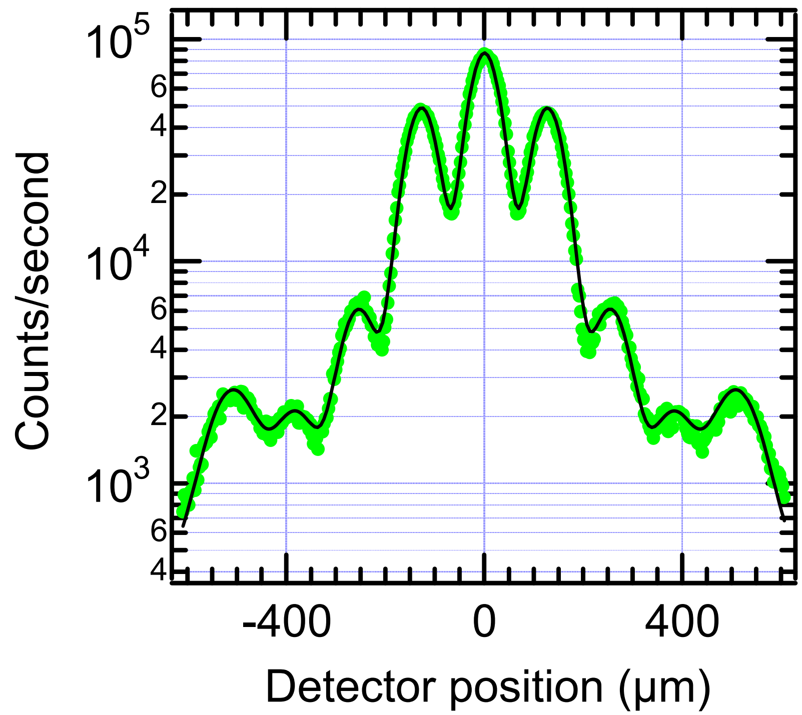
- Diffraction
  - Matter gratings
  - Light gratings
- Interferometers
- Internal state versions

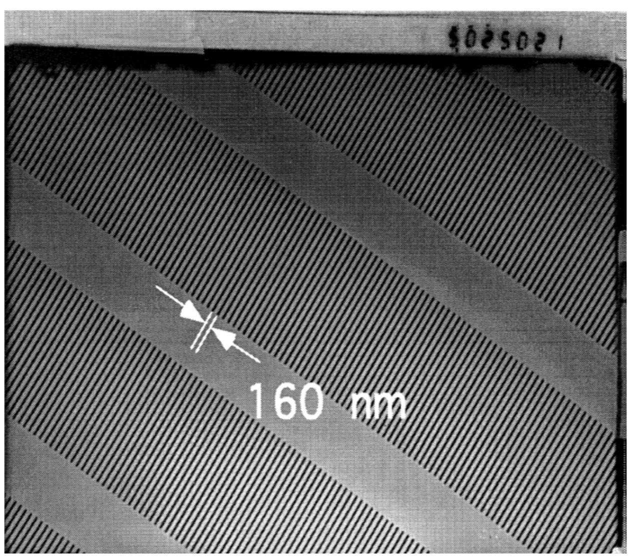


$$v_{sodium} = 1 \text{ km/s}$$

$$\lambda_{dB} = \frac{h}{mv} = 0.17 \text{ Å}$$

$$\theta_{\text{diffraction}} = \frac{\lambda_{\text{dB}}}{d} = 10^{-4} \text{ rad}$$

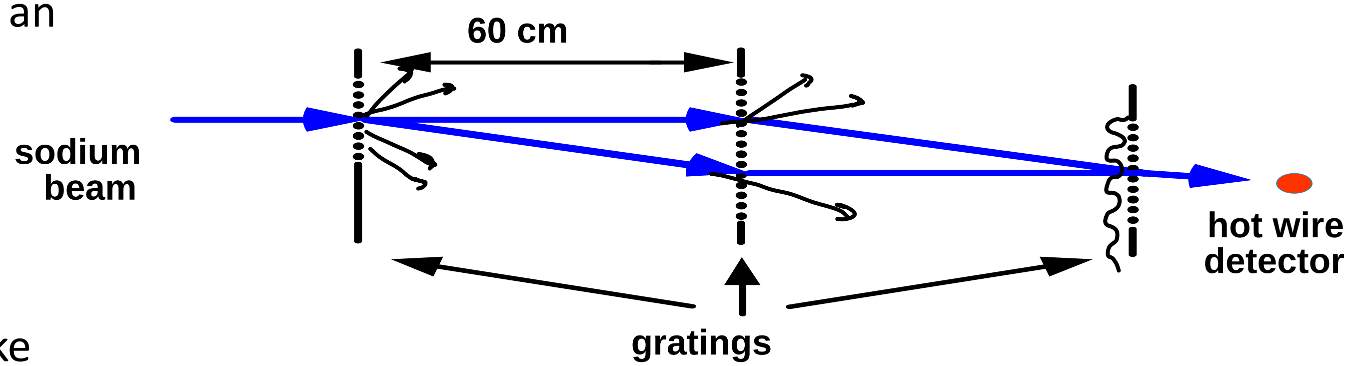




(MIT)

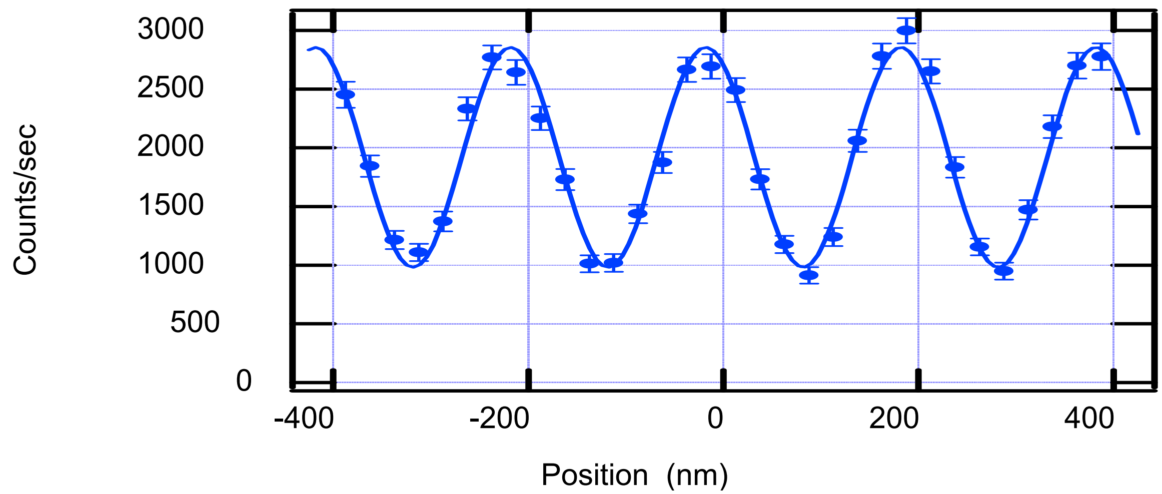
Atom Interferometer

- Use 3 gratings to make an interferometer



- The first 2 gratings make an atomic interference pattern at the location of the 3rd grating

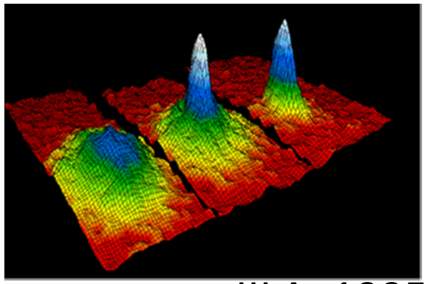
- The third grating is scanned and the atomic transmission is periodic



(MIT/Arizona)

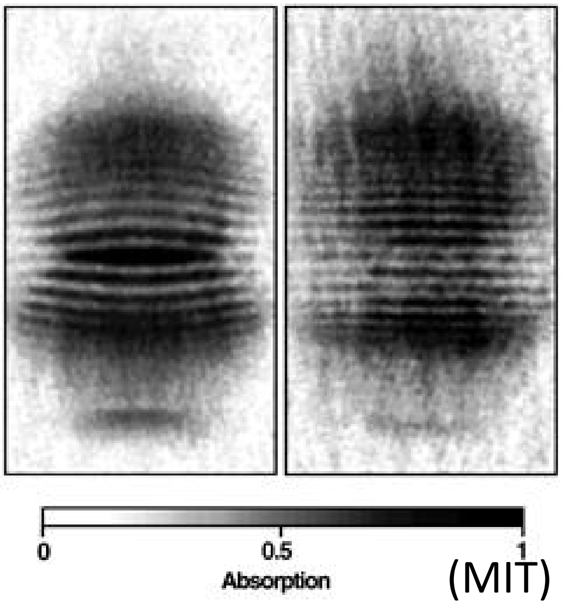
Coherent atom optics

- Enter Bose-Einstein condensation
- Atom lasers
- Coherence

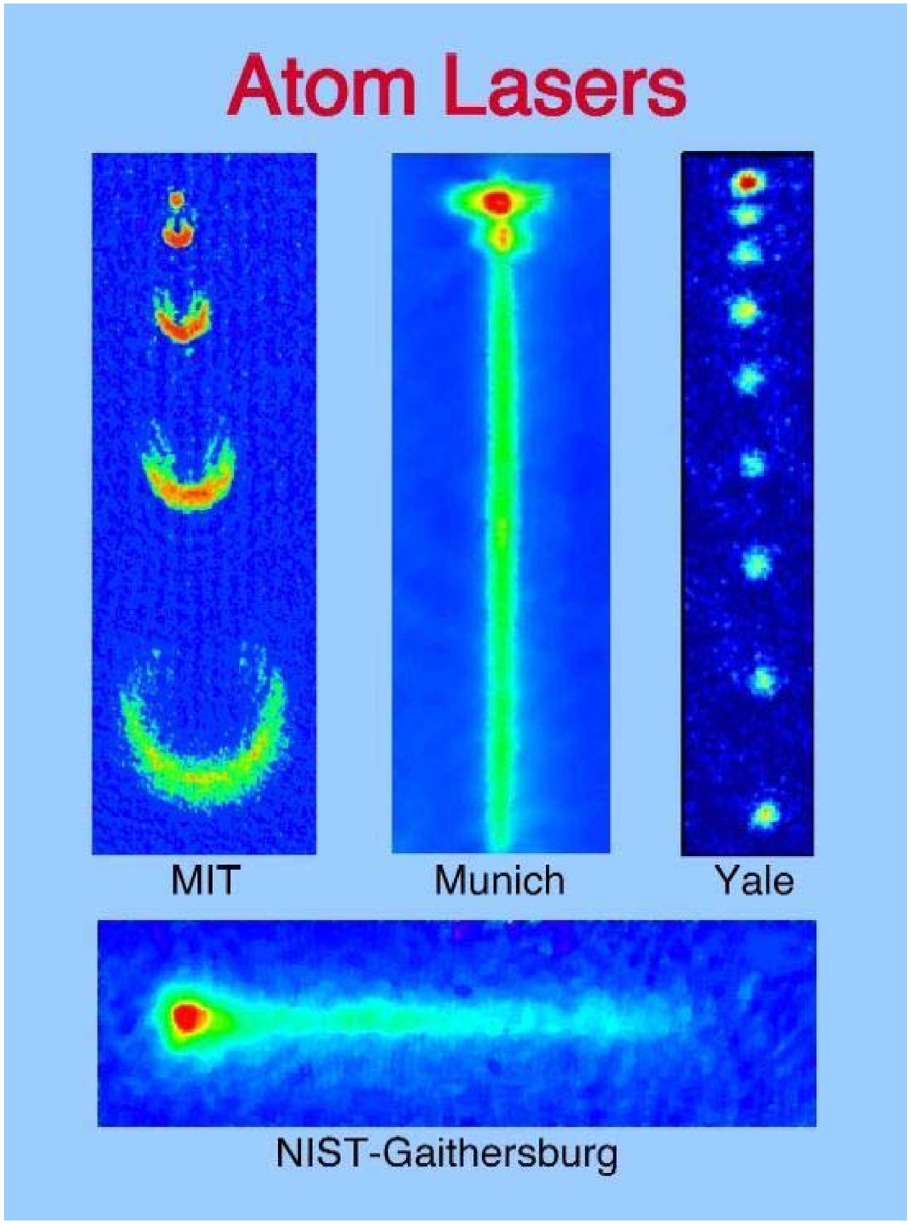


JILA, 1995

Interference of 2 BEC's (1997)

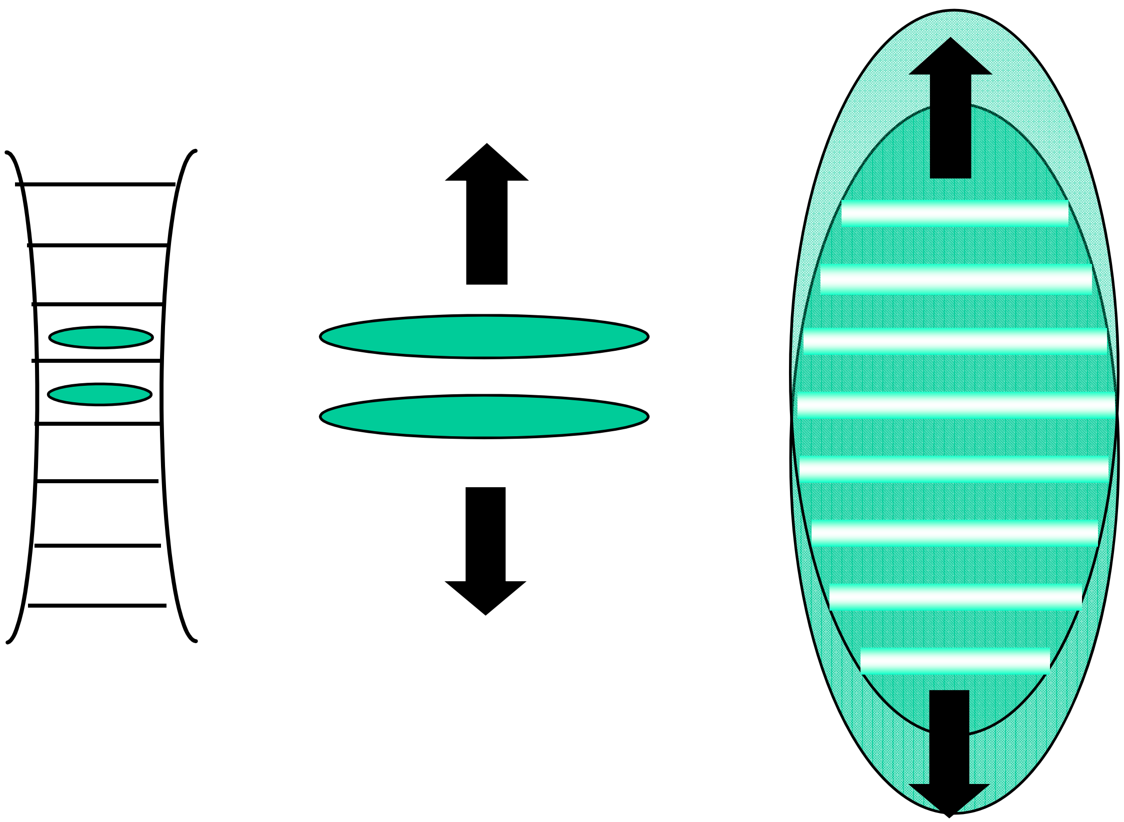


(MIT)

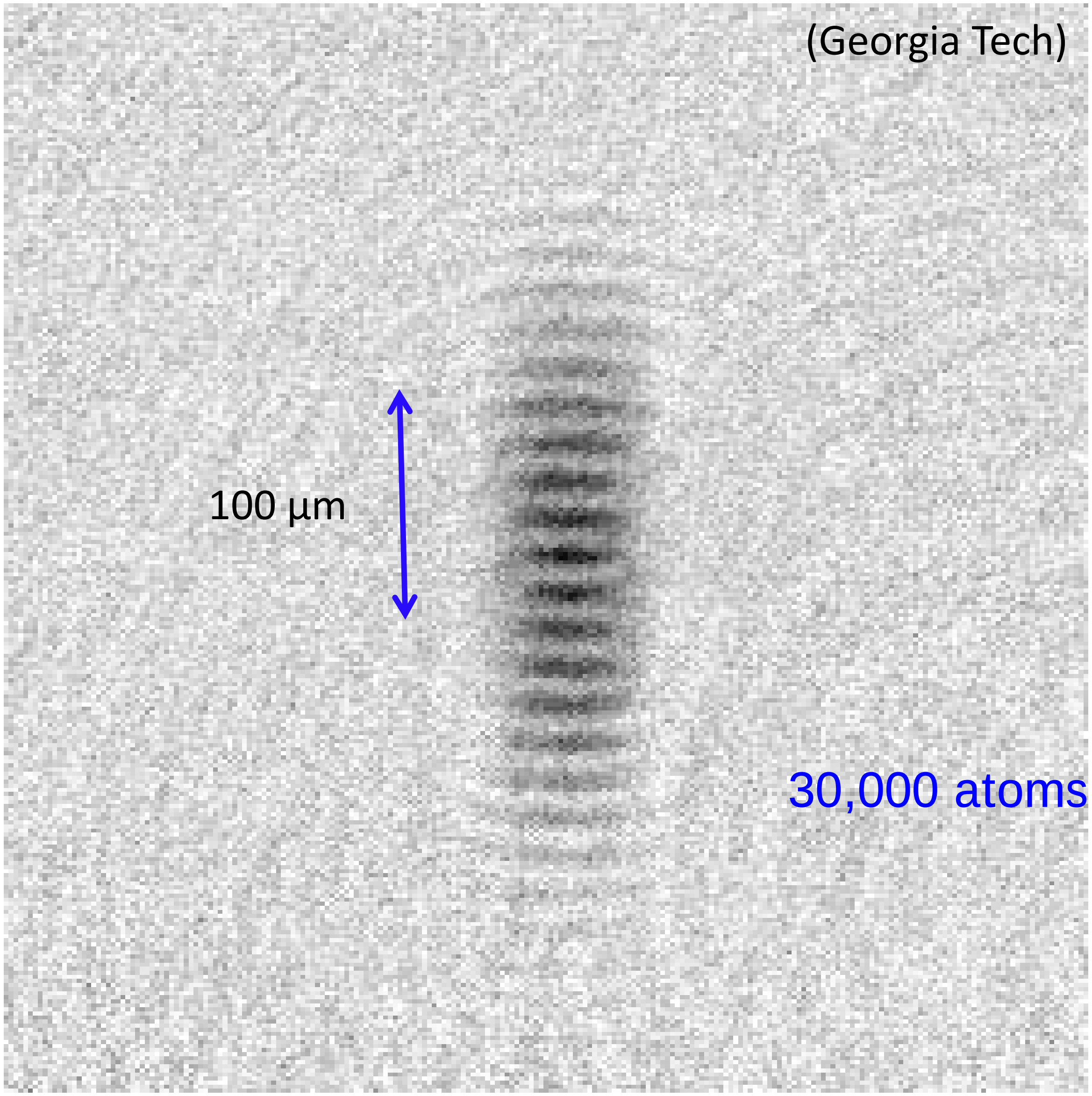


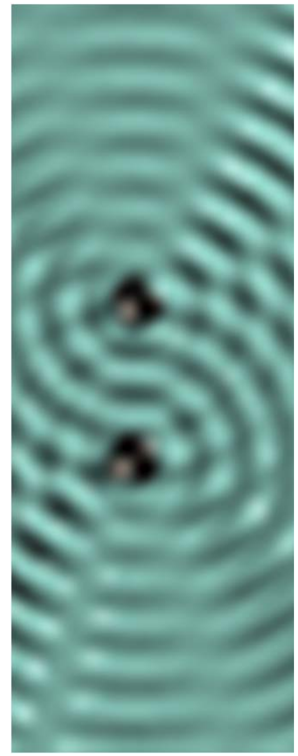
Coherence of condensates

- BEC's are the atomic analogues of coherent lasers
- If two condensates overlap during expansion, they will interfere
- Analogous to interference of two coherent independent lasers



Atomic interference of Bose condensed atoms





Non-linear atom optics

- Collision interactions are non-linear
- Gross-Piteavski equation
  - Non-linear Schrodinger equation

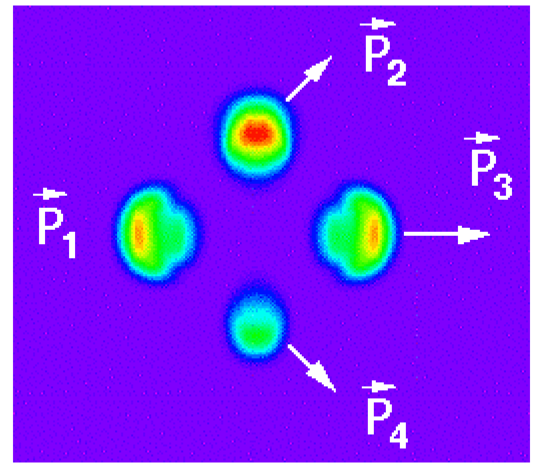
$$i\hbar \frac{\partial \Psi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + g\,|\Psi(\mathbf{r},t)|^2 \right)\Psi(\mathbf{r},t)$$

- Very strong non-linearities
  - Minimal loss

$$g = \frac{4\pi\hbar^2 a_s}{m}$$

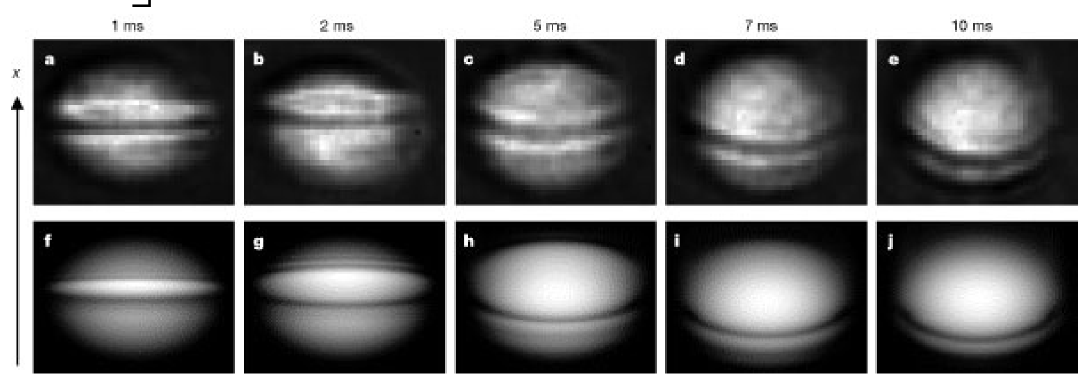
- Atomic four-wave mixing
  - External (motional) states
  - Internal (spin) states

$$H = \int dx \left[\frac{1}{2}\partial_x\psi^\dagger \partial_x\psi + g\psi^\dagger\psi^\dagger\psi\psi \right]$$



(NIST)

- Atomic solitons

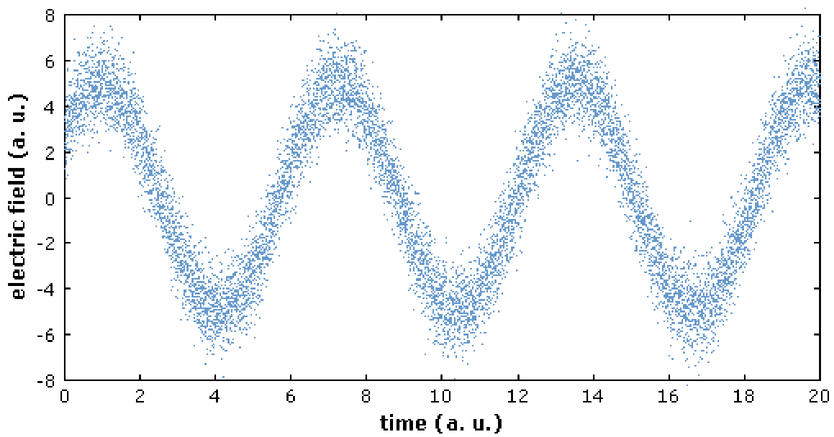


Quantum atom optics

- Squeezed states, entangled states
  - Enhanced sensor performance
    - Beyond the Standard Quantum Limit
    - 'shot-noise' limit
    - Projection noise limit
- Entangled states
  - Continuous variable quantum information
- Beyond coherent atom wave ('mean field') approximation

- It is all about the fluctuations

$$H = \int dx \left[\frac{1}{2} \partial_x \psi^\dagger \partial_x \psi + g \psi^\dagger \psi^\dagger \psi \psi \right]$$



Why atom optics?

- Atomic sensors
  - Practical applications
    - Frequency metrology
    - Inertial metrology
    - Field sensors
- Applications to fundamental physics
  - Variation of fundamental constants
  - Fundamental symmetries
    - Parity violation
    - Permanent electric dipole moments
- Quantum information
- Quantum many-body physics
  - simulators

Atomic sensors

- Clocks
- Accelerometers
- Gyroscopes
- Magnetometers

Frequency is the most accurately measured quantity.

– We are really good at counting

Winning strategy

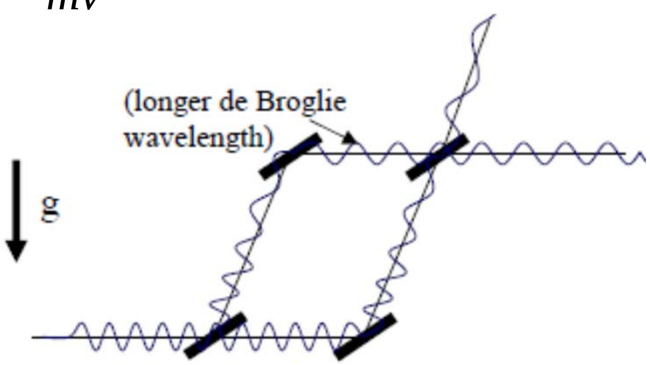
– Convert your measurement to a frequency measurement

deBroglie wave interferometers

Accelerations/gravity

--atom slows in climbing potential

$$\lambda_{dB} = \frac{h}{mv}$$



$$\Delta\varphi_{accel} = \frac{gt^2}{\lambda}$$

g : acceleration

t : transit time

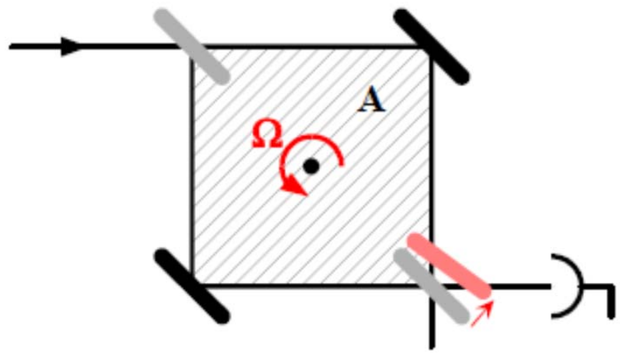
λ : wavelength of light

Measuring free-fall with a nanometer ruler

deBroglie wave interferometers

Rotations/Gyroscopes

--Sagnac effect for deBroglie waves



$$\Delta\varphi_{Sagnac} = \frac{4\pi A\Omega}{\lambda v}$$

Ω : rotation rate

A : enclosed area

v : speed of wave

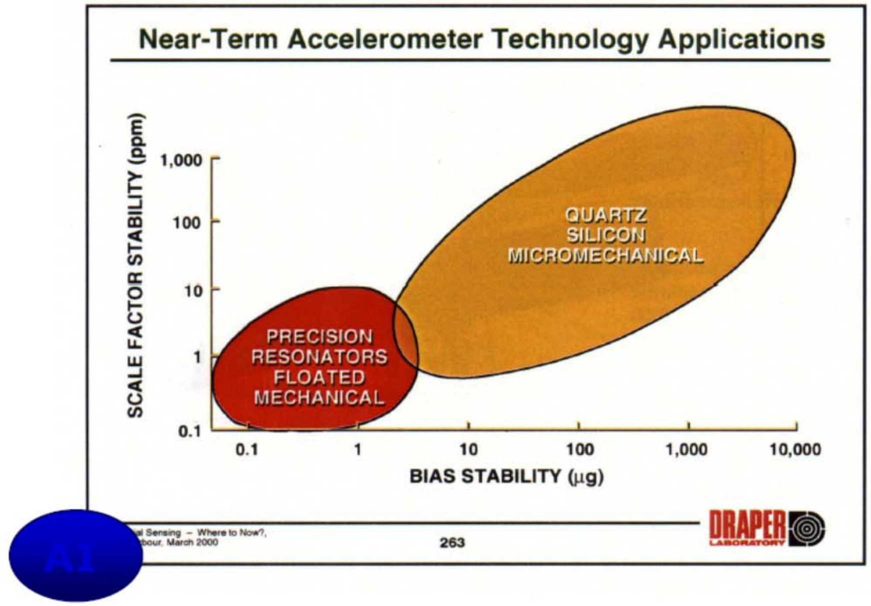
$$\frac{\Delta\varphi_{atom}}{\Delta\varphi_{light}} = \frac{\lambda_{light}c}{\lambda_{deBroglie}v}$$

$$= \frac{m_{atom}c^2}{h\nu} > 10^{10}$$

Atomic inertial guidance (Kasevich)

Light-puse AI accelerometer characteristics

- Bias stability: $<10^{-10}$ g
- Noise: 4×10^{-9} g/Hz$^{1/2}$
- Scale Factor: 10^{-12}



Light-puse AI gyroscope characteristics

- Bias stability: <60 μdeg/hr
- Noise (ARW): 4 μdeg/hr$^{1/2}$
- Scale Factor: <5 ppm

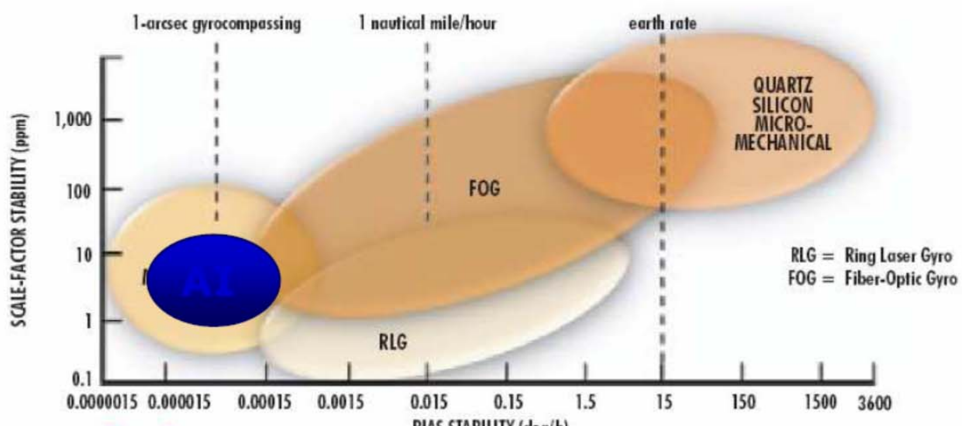
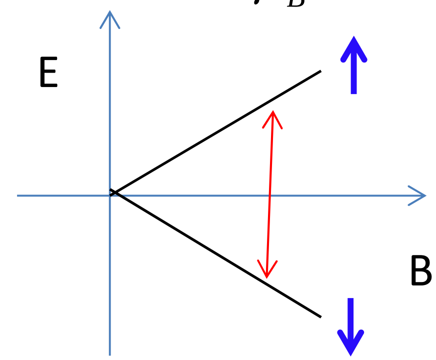


Figure 8

Source: Proc. IEEE/Workshop on Autonomous Underwater Vehicles

Atomic magnetometers

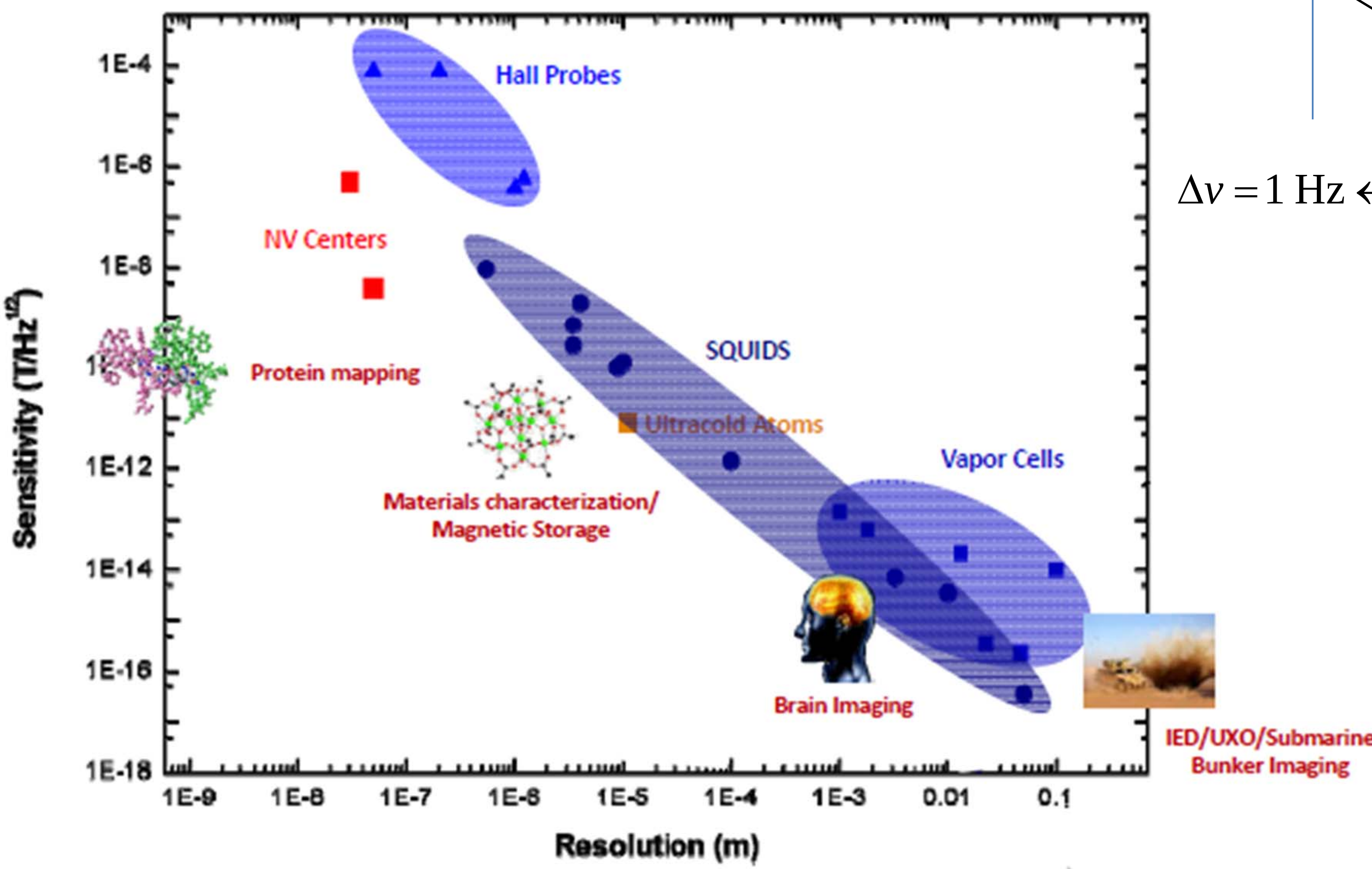
$$E = h\nu = \mu_B \cdot B$$



$$\Delta\nu = 1\ \mathrm{Hz} \leftrightarrow \ \Delta B = 10^{-10}\ \mathrm{T}$$

Frequency metrology of magnetic Zeeman shifts

--Measure Larmor precession of atomic electron spin



Fundamental measurements

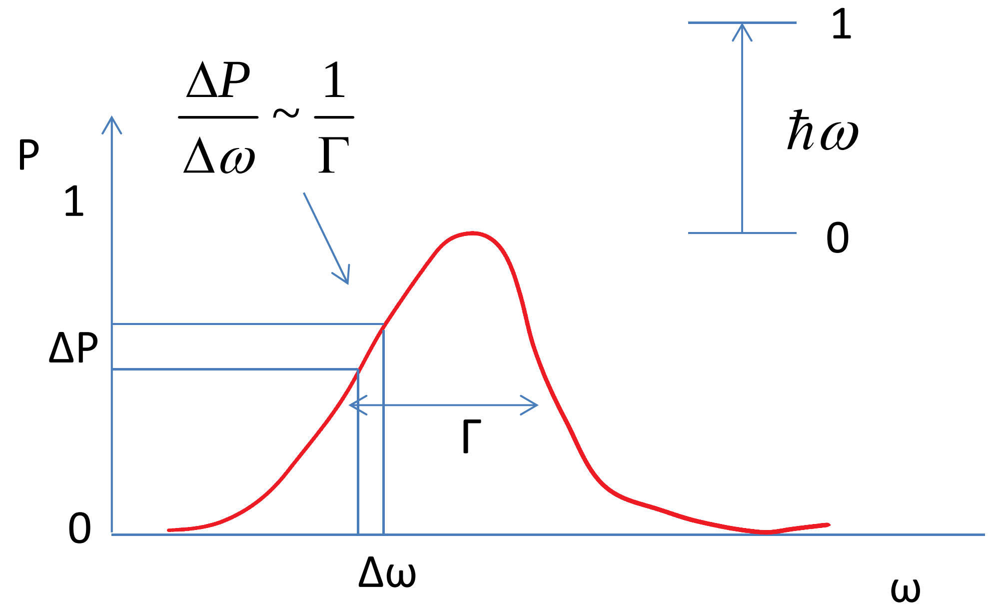
- Gravitational
  - Equivalence principle
  - Tests of post-Newtonian gravity
- Variation of fundamental constants
  - Is the fine structure constant indeed constant
- Tests of fundamental symmetries
    - Parity violation
    - Permanent electric dipole moments

Measurement precision

- Usual case for precision measurement → two main elements

- Intrinsic width of feature to be measured
  - Linewidth, Γ

- Ability to resolve the line
  - Relative precision
  - Splitting the line



Measurement precision:

$$\Delta\omega = \Delta P\,\Gamma$$

Quantum projection noise (QPN)

- Consider a 2-level atom prepared in the superposition state

$$|\phi\rangle = c_+ |+\rangle + c_- |-\rangle$$

- To evaluate the probability for measuring the atom in the upper level, we use the projection operator

$$\hat{p}_+ = |+\rangle\langle+|$$

$$p_+ = \langle p_+ \rangle = \left|\langle\phi|\,\hat{p}_+\,|\phi\rangle\right|^2 = \left(c_+^* \langle+| + c_-^* \langle-|\right)|+\rangle\langle+|\left(c_+ |+\rangle + c_- |-\rangle\right) = |c_+|^2$$

$$\left\langle p_+^2 \right\rangle = \left|\langle\phi|\,\hat{p}_+^2\,|\phi\rangle\right|^2 = \left(c_+^* \langle+| + c_-^* \langle-|\right)\left(|+\rangle\langle+|\right)^2\left(c_+ |+\rangle + c_- |-\rangle\right) = p_+$$

- The uncertainty is:

$$\Delta p_+ = \sqrt{\left\langle p_+^2 \right\rangle - \left\langle p_+ \right\rangle^2} = \sqrt{p_+\left(1 - p_+\right)}$$

Quantum projection noise (QPN)

$$|\phi\rangle = c_+|+\rangle + c_-|-\rangle$$

- Uncertainty in finding the atom in upper state:

$$\Delta p_+ = \sqrt{p_+(1-p_+)} \qquad p_+ = |c_+|^2$$

- This is zero for the atom prepared in the upper or lower state, and has a maximum value for an equal superposition

$$\Delta p_+ = \tfrac{1}{2} \quad \text{for} \quad p_+ = \tfrac{1}{2}$$

- If we have *N* identically prepared atoms that are <u>non-interacting</u>, then the probability for detecting N_+ atoms in the upper state is:

$$P(N_+, N, p_+) = \frac{N!}{N_+!(N-N_+)!}(p_+)^{N_+}(1-p_+)^{(N-N_+)}$$

- With an uncertainty:

$$\Delta P_+ = \sqrt{N}\sqrt{p_+(1-p_+)} \Rightarrow \Delta P_+ = \frac{\sqrt{N}}{2} \quad \text{for} \quad p_+ = 1/2$$

- The relative uncertainty is:

$$\Delta P_+' = \frac{\sqrt{N}}{2}\frac{2}{N} = \frac{1}{\sqrt{N}}$$

Also known as the Standard Quantum Limit (SQL) or “shot-noise” limit (SNL)

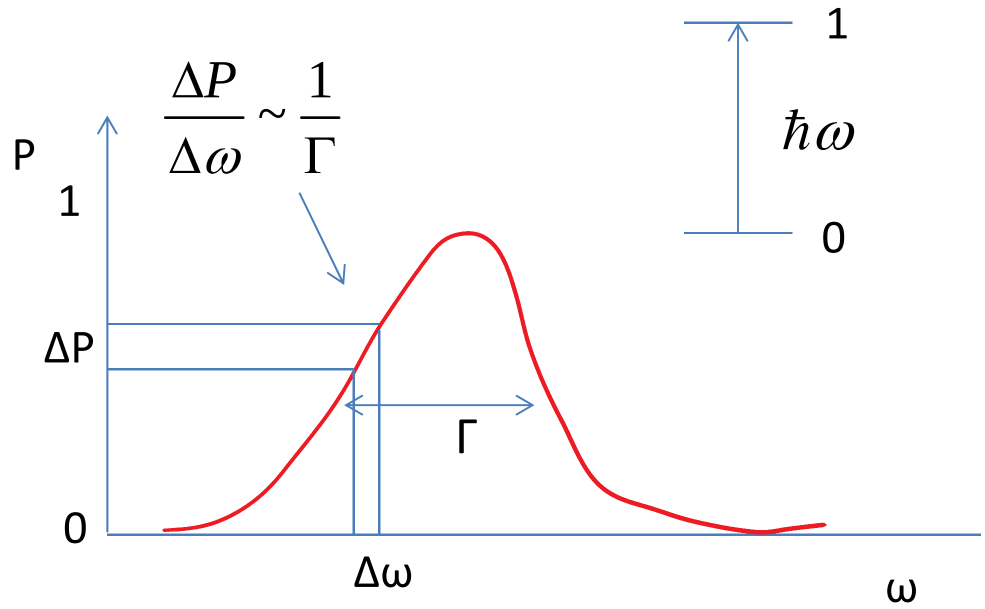
Measurement precision

- Usual case for precision measurement → two main elements

- Intrinsic width of feature to be measured
  - Linewidth, Γ

- Ability to resolve the line
  - Relative precision
  - Splitting the line

- In order to increase precision:
  - Decrease linewidth
    - Narrow intrinsic line
    - If Fourier limited, use longer measurement time
  - Increase N
    - Number of particles/trials
  - Or…do not accept the SQL!!



Measurement precision:

$$\Delta\omega = \Delta P \, \Gamma = \frac{\Gamma}{\sqrt{N}}$$

25 years of squeezed light

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Quantum-mechanical noise in an interferometer

Carlton M. Caves
W. K. Kellogg Radiation Laboratory, California Institute of Technology, Pasadena, California 91125

- First demonstrated in 1984 by Slusher et al.
  - Squeezing in the complex plane U(1)
- Recently reached the 12 dB milestone in lab
- Currently implemented in gravitation interferometers
  - GEO, LIGO

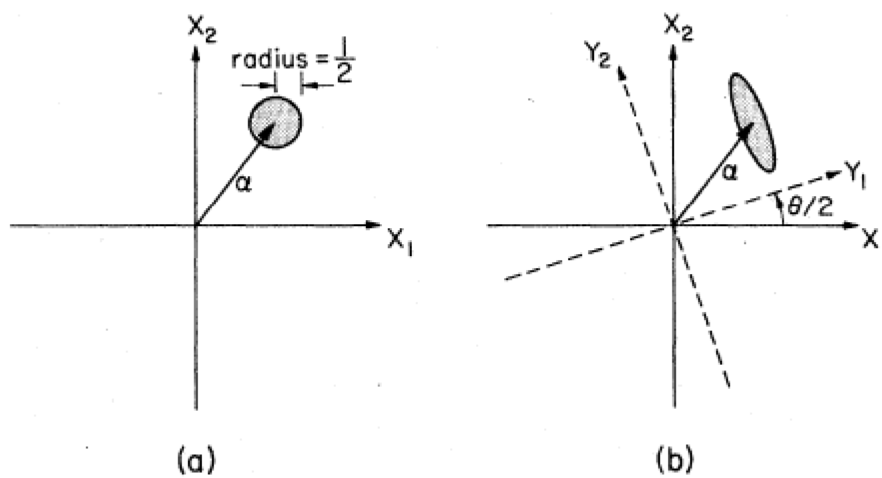


FIG. 1. (a) Error circle in complex-amplitude plane for coherent state $|\alpha\rangle$. (b) Error ellipse in complex-amplitude plane for squeezed state $|\alpha, re^{i\theta}\rangle$ $(r>0)$.

$$X_1 + iX_2 = a$$

$$N = a^\dagger a$$

Why atomic squeezed states?

- To go beyond SQL limited precision towards Heisenberg-limited precision
- Enhanced metrological precision
  - In many experiments, the linewidth of the measured transition is Fourier limited
    - Very narrow hyperfine transitions
    - Interaction time-limited
  - Cold atoms have revolutionized the field
    - Millisecond measurement times → seconds
  - The cost: can make atoms really cold, but not very many at a time
    - $1 < N < 10^6$

$$\Delta\omega = \frac{\Gamma}{\sqrt{N}} \to \frac{\Gamma}{N}$$

SQL → Heisenberg limit

- Also
  - Continuous variable quantum information
  - Explore non-equilibrium quantum many-body physics

Uncorrelated (unsqueezed) spin-1/2 particles

– For a collection of *N* 2-level systems, we can define collective spin operators

$$\mathbf{S} = \sum_i \mathbf{s}_i$$

– These satisfy the usual angular momentum commutation rules

$$\left[S_i, S_j\right] = i\varepsilon_{ijk} S_k$$

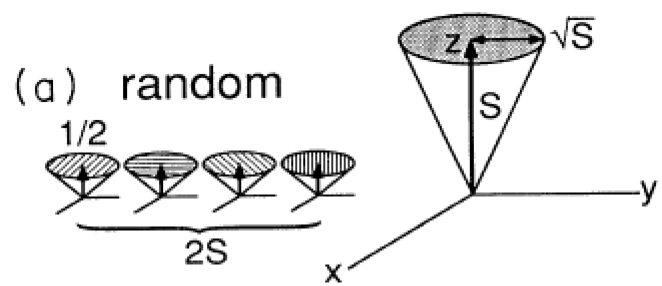
– And obey the uncertainty relationship

$$(\Delta S_i)^2 (\Delta S_j)^2 \geq \tfrac{1}{4}\langle S_k \rangle^2$$

– Eigenstates of S(θ,φ) are coherent spin states

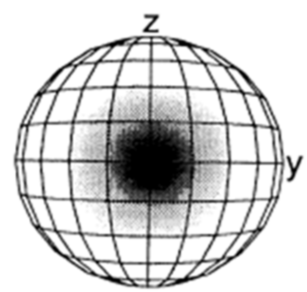
$$\left|\theta,\phi\right\rangle = \prod_i \left(\cos\tfrac{\theta}{2}\left|+\right\rangle + e^{i\phi}\sin\tfrac{\theta}{2}\left|+\right\rangle\right)_i$$

- Represented on a Bloch sphere (θ, φ)
- Product states, uncorrelated
- Minimum uncertainty states
  - Uncertainty circle



$$S = \frac{N}{2}$$

$$\Delta S_x = \Delta S_y = \frac{\sqrt{N}}{2}$$



(a)

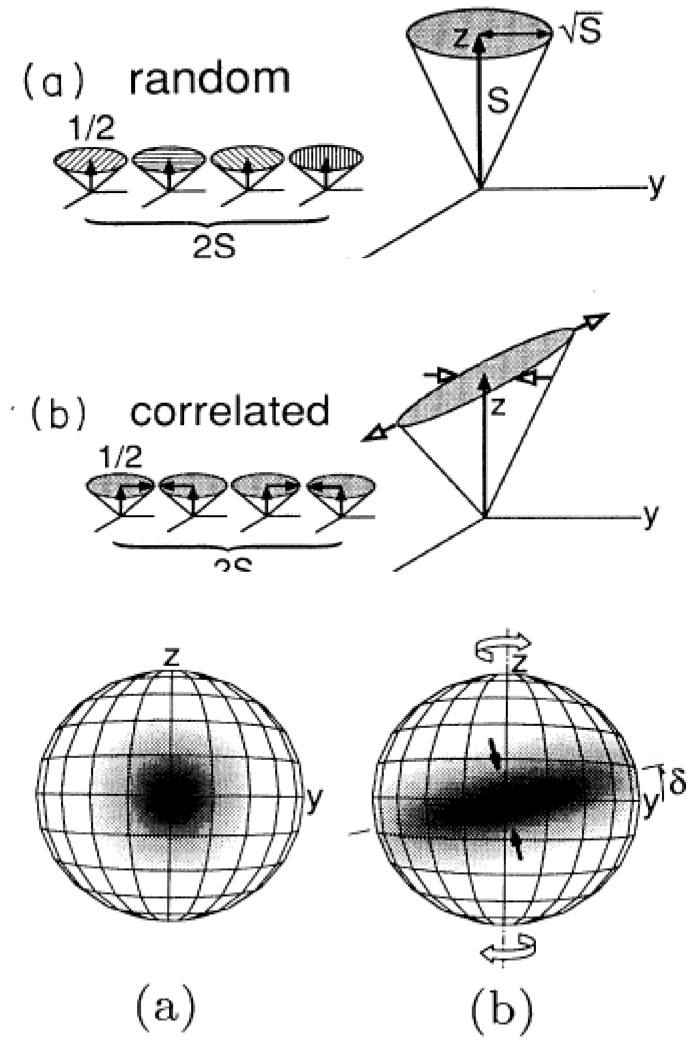
Spin-1/2 squeezed states

- Spin squeezed states (SSS) have a variance of a spin component normal to the mean spin vector that less than the SQL
- Squeezing on the Bloch sphere: SU(2)
- Squeezing parameter (there are a few)

$$\xi^2 = \frac{2(\Delta S_\perp)^2}{\left|\left\langle S_{\theta,\phi}\right\rangle\right|} < 1$$

- Measurement improvement:

$$\Delta\phi = \frac{\xi}{\sqrt{N}}$$



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Squeezed spin states

Masahiro Kitagawa and Masahito Ueda
Nippon Telegraph and Telephone Corporation Basic Research Laboratories, Musashino, Tokyo 180, Japan
(Received 12 February 1991; revised manuscript received 3 December 1992)

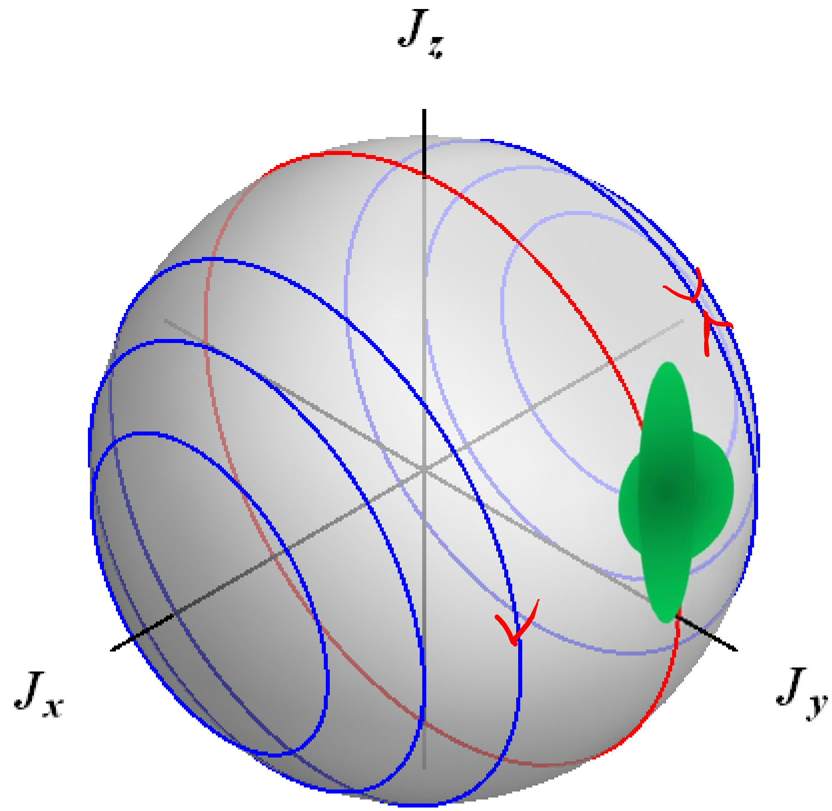
Squeezing Hamiltonians

(notation change S→J)

- Single axis twisting
  - Grab plus and minus x-axis and twist in opposite direction
- 2 elliptical fixed points (stable)

- Red contours are infinite period
- Blue contours are positive energy
- Green contours are negative energy

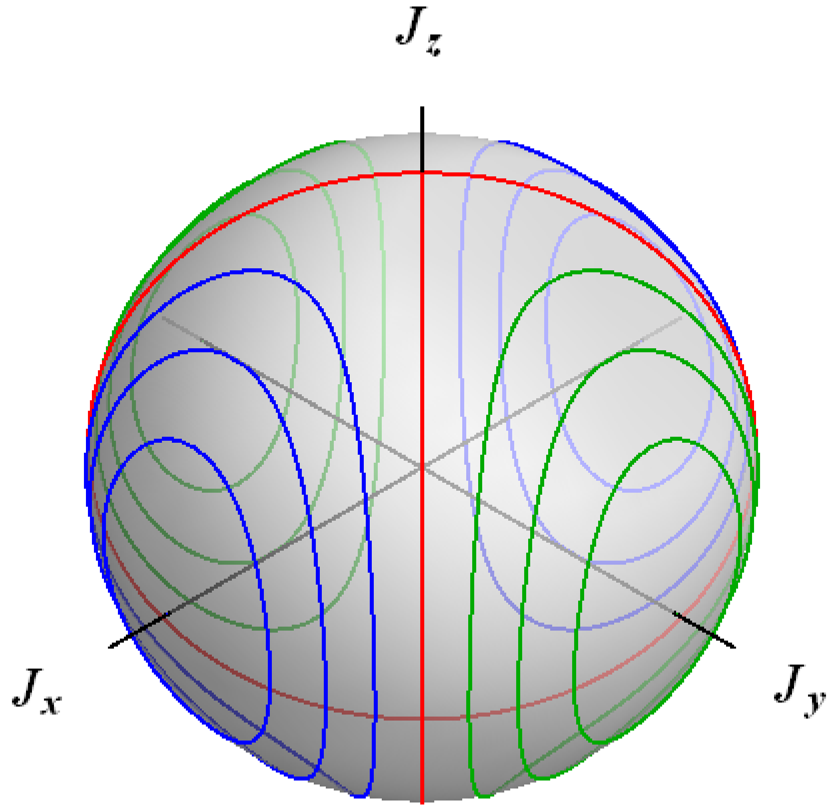
$$H = J_x^2$$



Squeezing Hamiltonians

$$H = J_x^2 - J_y^2$$

- Two-axis, counter-twisting
  - X = Twist,
  - Y = CounterTwist
- 2 hyperbolic fixed points
- 4 elliptical fixed points
  - (2 stable, 2 unstable)



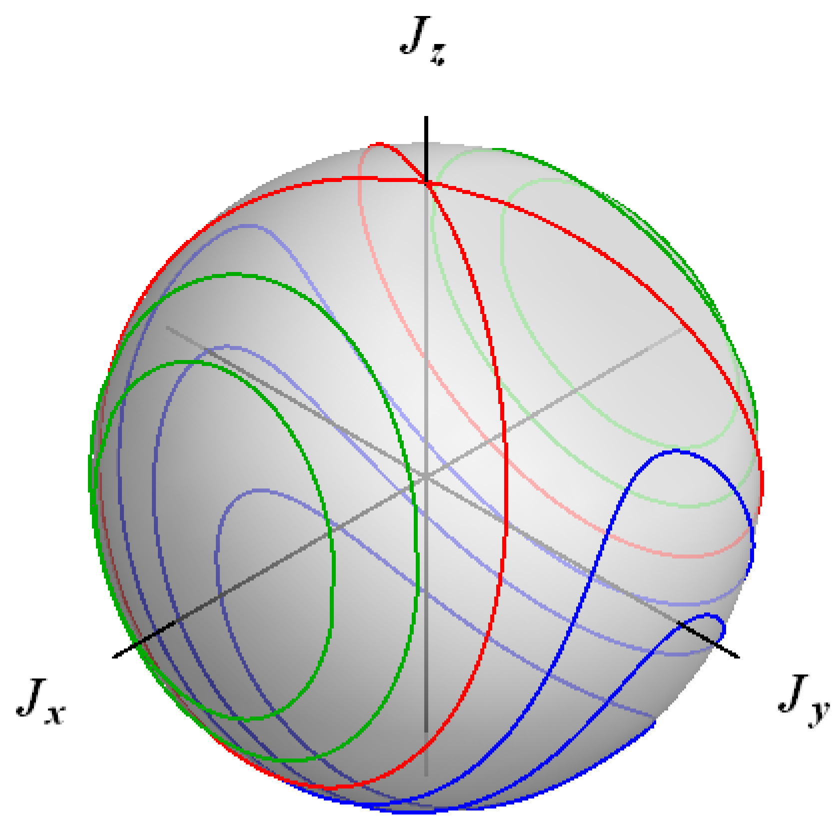
- Red contours are infinite period
- Blue contours are positive energy
- Green contours are negative energy

Squeezing Hamiltonians

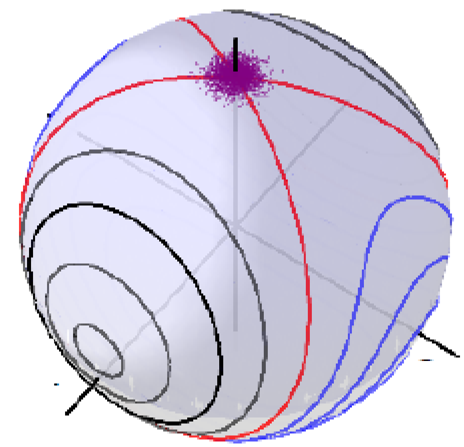
- Single axis twisting+rotation
  - X = CounterTwist
  - Z = Rotate
- 1 hyperbolic fixed point
- 3 elliptical fixed points
  - (2 stable, 1 unstable)

- Red contours are infinite period
- Blue contours are positive energy
- Green contours are negative energy

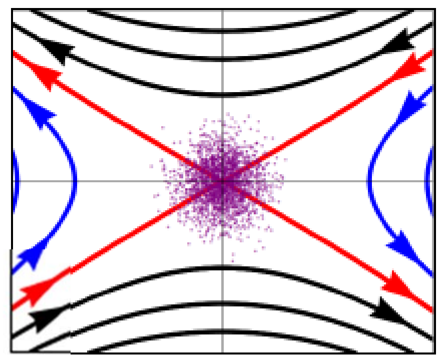
$$H = -J_x^2 + J_z$$

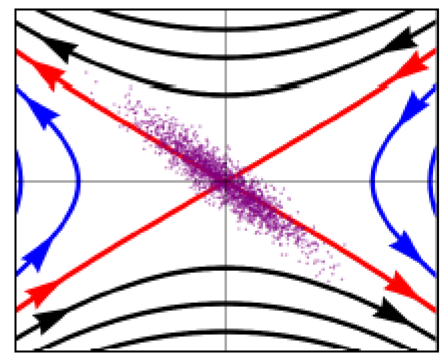


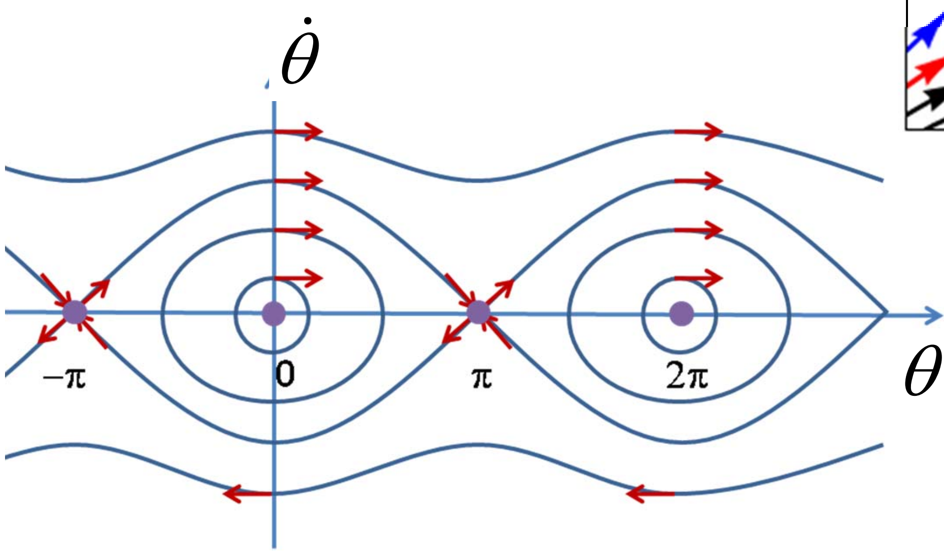
Squeezing at a hyperbolic fixed point



- Squeezing is a result of motion on the converging branch of separatrix
- State is anti-squeezed along diverging branch of separatrix
- Hyperbolic fixed point on sphere is locally identical to inverted pendulum

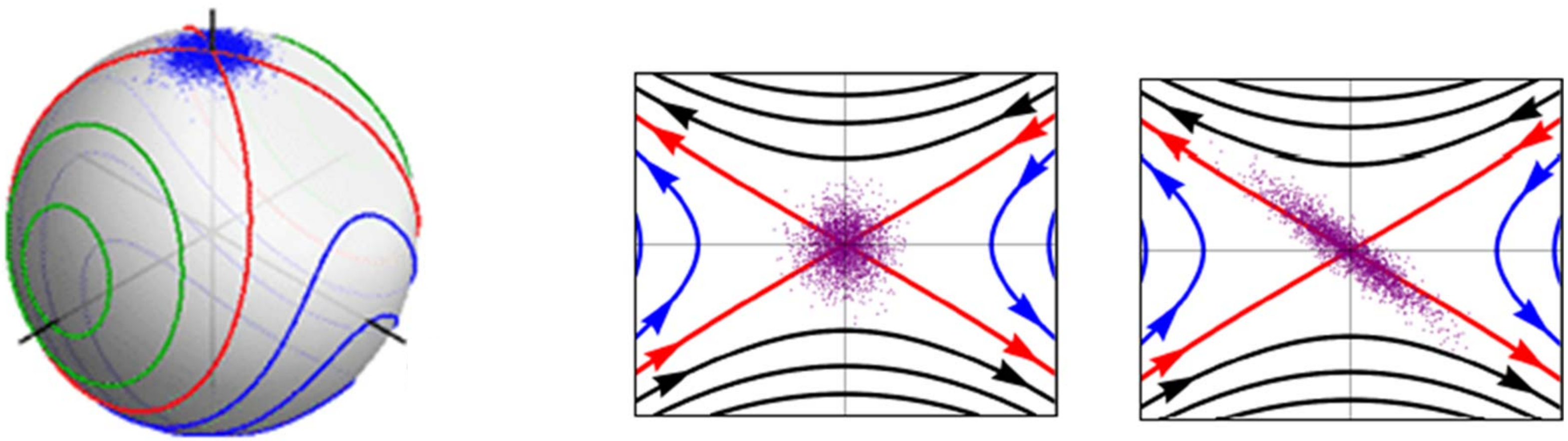






$$H = -J_x^2 + J_z$$

Squeezing at a hyperbolic fixed point



Sources of atom squeezing

- Generating squeezing generally requires non-linear interactions
- There are two main approaches

- Atom-light interactions
  - Transfer of squeezed states from light to atoms
  - Non-linear atom-light interactions in optically dense samples

  (Hammerer, Sorensen and Polzik, RMP 2010)

- Atom-atom interactions
  - Collisional interactions in a BEC

  (Nori et al., Phy. Rep. 2011)

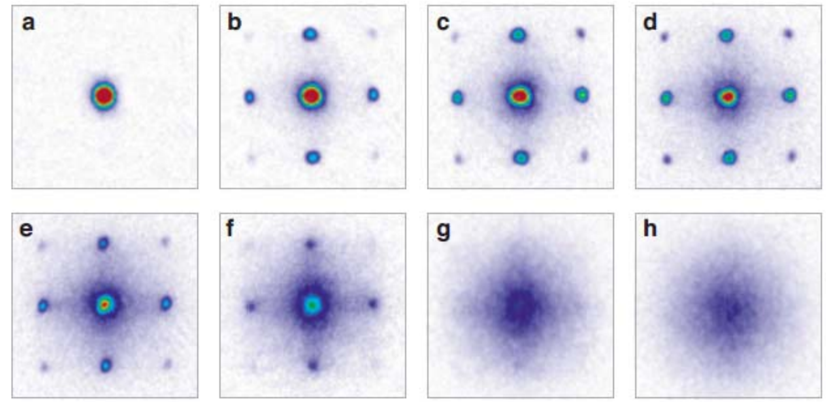
$$H = \int dx \left[\frac{1}{2} \partial_x \psi^\dagger \partial_x \psi + g \psi^\dagger \psi^\dagger \psi \psi \right]$$

$$\hat{H}_{4WSM} = \lambda \left(\hat{a}_0^2 \hat{a}_{+1}^\dagger \hat{a}_{-1}^\dagger + \hat{a}_0^{\dagger 2} \hat{a}_{+1} \hat{a}_{-1} \right)$$

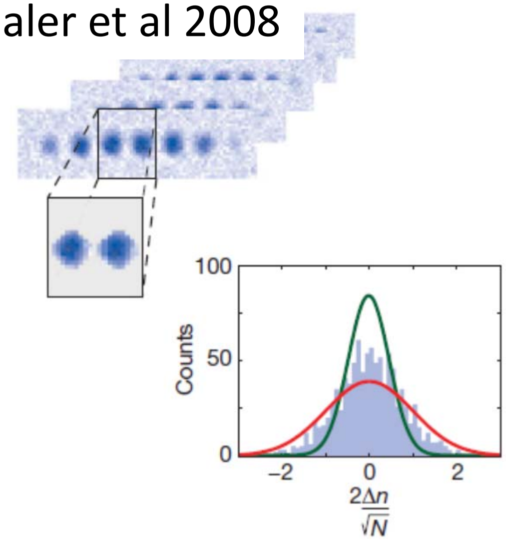
Quantum atom optics experiments

- Number squeezing
  - Lattice condensates
    - Relative fluctuations in site-to-site atom population
  - Bose-Hubbard physics
  - Mott insulator is extreme case
    - 1 atom/lattice site
  - Detected indirectly in interference
  - Detected directly in large period lattice
- Correlated atom pairs
  - Molecular dissociation
  - Colliding condensates
  - Spin mixing in a spin-1 condensate
- Quadrature squeezing
  - Psuedo-spin-1/2 system
  - Spin-1 system

Greiner et al 2002



Oberthaler et al 2008

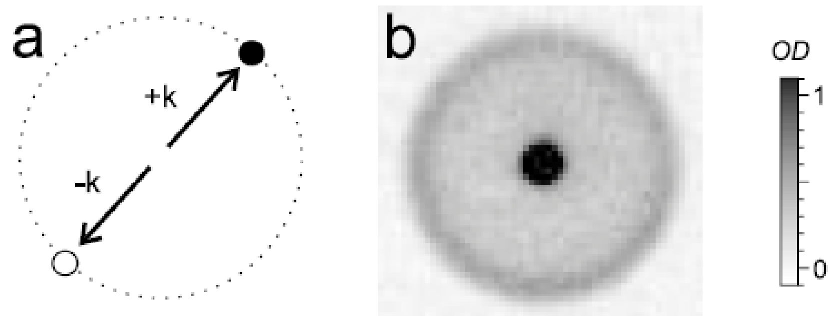


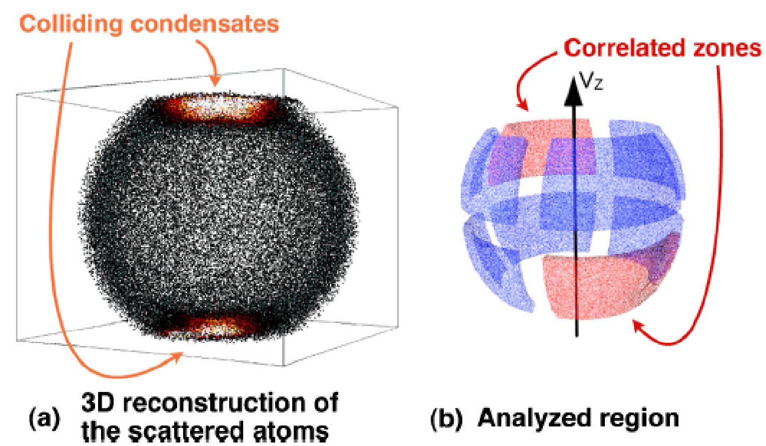
Correlated atoms

- Momentum correlated atom
  - Dissociating ultracold molecules
  - Colliding condensates
- Spin-correlated atoms
  - Spin-changing collisions constrained by angular momentum conservation
  - Observed in spin-1 systems 2010-11
    - Georgia Tech, Hannover, Heidleberg

$$\hat{H}_{4WSM} = \lambda(\hat{a}_0^2 \hat{a}_{+1}^\dagger \hat{a}_{-1}^\dagger + \hat{a}_0^{\dagger 2} \hat{a}_{+1} \hat{a}_{-1})$$

Jin 2005





Westbrook 2010

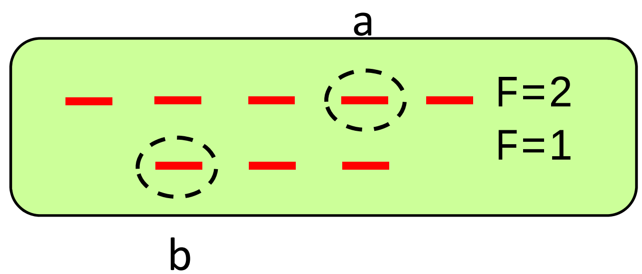
Quadrature squeezing in a two-component system (pseudospin-1/2)

- Useful for interferometry
- Uses a 2-compenent BEC
- Single axis twisting with rotations

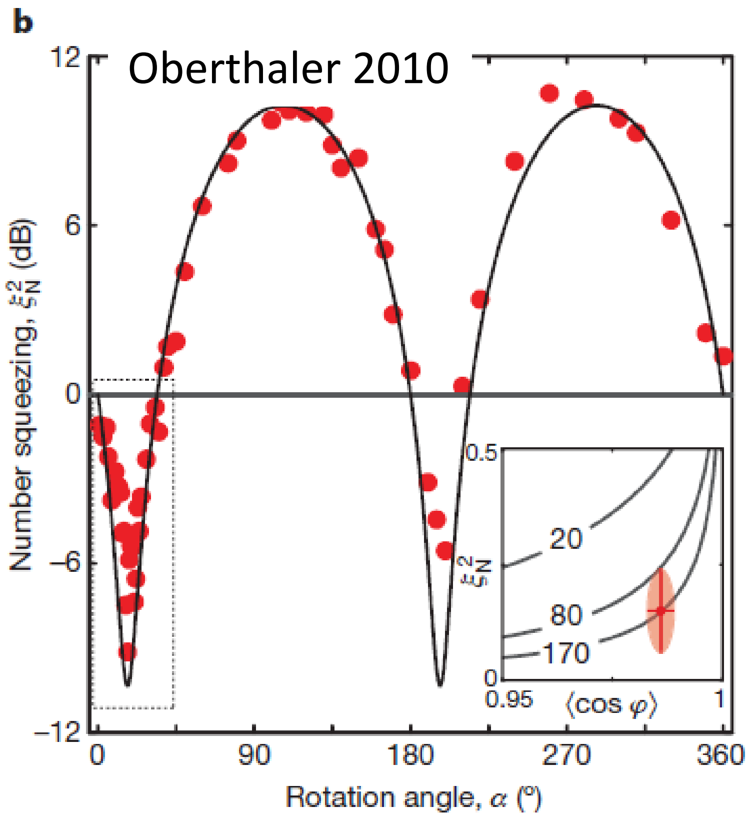
$$H/\hbar = \Delta\omega_0 J_z + \chi J_z^2 + \Omega J_\phi$$

- Collisional interaction non-linearity

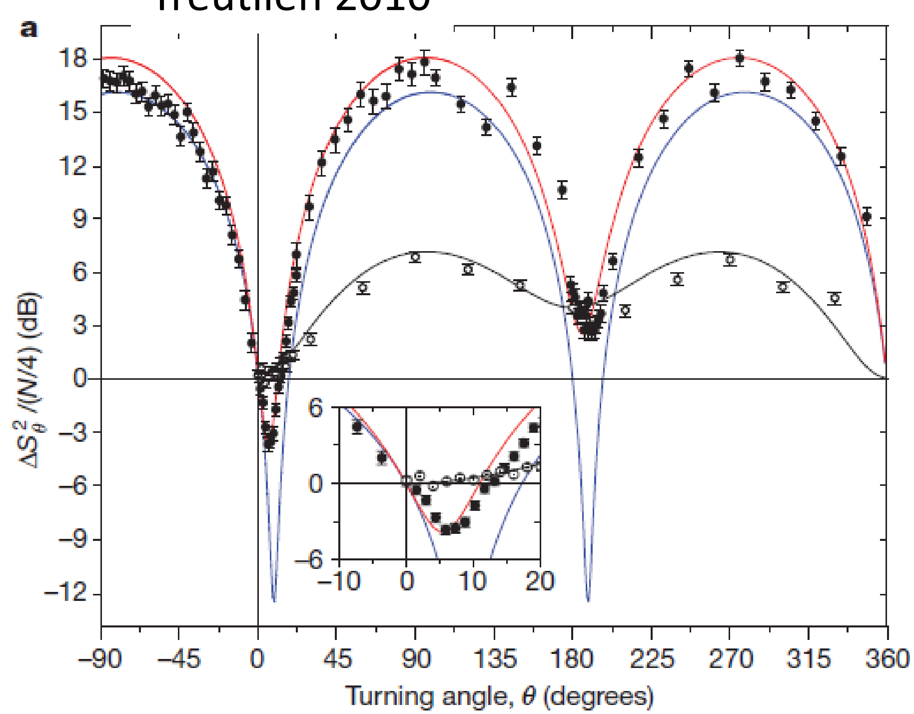
$$\chi \propto a_{aa} + a_{bb} - 2a_{ab}$$



Requires controlling χ using Feshbach resonance or control of spatial overlap



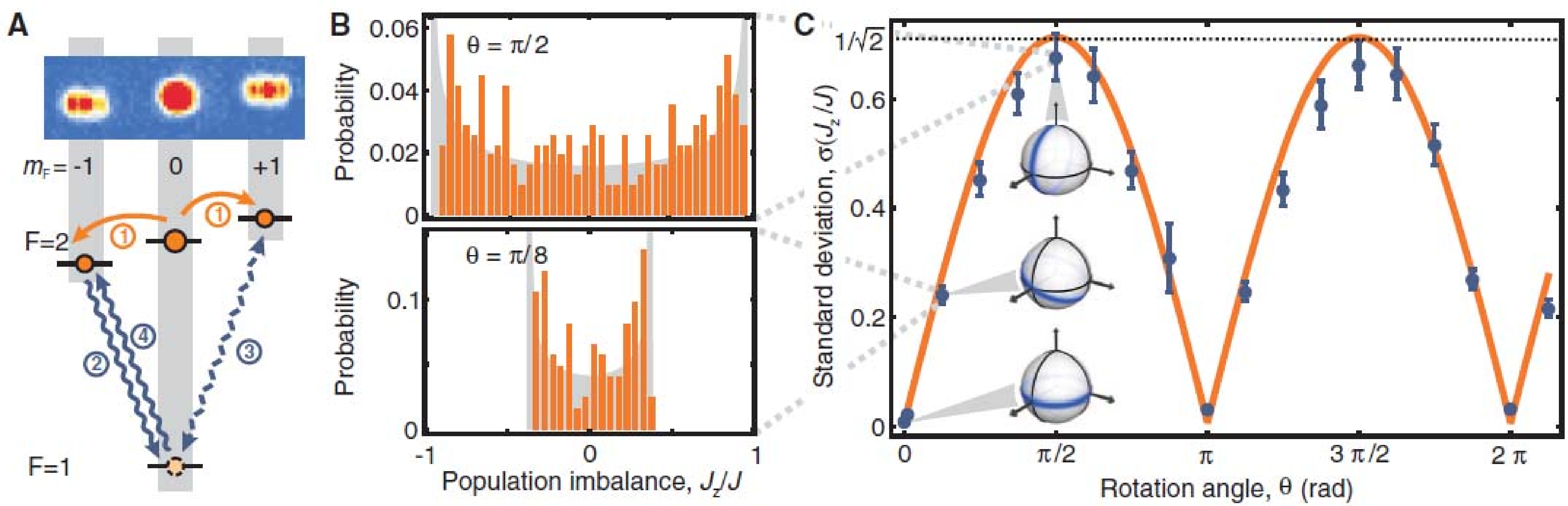
Treutlien 2010



Squeezing in a three-component system (spin-1)

- Demonstrated by 2 groups, 2011
  - Hannover
    - Spin-1 interactions, then reduction to 2-state system, 1.6 dB squeezing
  - Georgia Tech
    - 8-10 dB squeezing observed
- Related work in Heidelberg
  - Atomic homodyne detection in a spin-1 system. Detection noise too high to observe squeezing

$$\hat{H}_{4WSM} = \lambda(\hat{a}_0^2\hat{a}_{+1}^{\dagger}\hat{a}_{-1}^{\dagger} + \hat{a}_0^{\dagger 2}\hat{a}_{+1}\hat{a}_{-1})$$



Quantum atom optics with Bose condensates
Part 2

Prof. Michael Chapman
Georgia Tech
Atlanta, GA

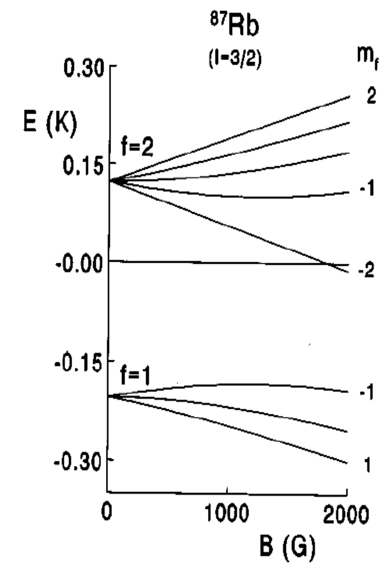
Spin-1 condensates

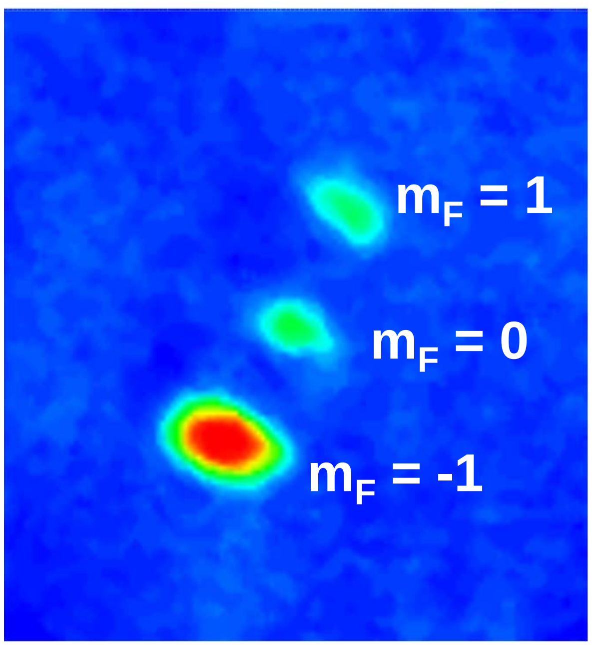
- BEC with vector order parameter
- Josephson dynamics in internal degrees of freedom
- 4-wave mixing
- Entanglement and squeezing

F=1

$m_F =$ -1 $\quad 0$ $\quad 1$

$$\vec{\phi}(\vec{r}) = \begin{pmatrix} \phi_1(\vec{r}) \\ \phi_0(\vec{r}) \\ \phi_{-1}(\vec{r}) \end{pmatrix}$$





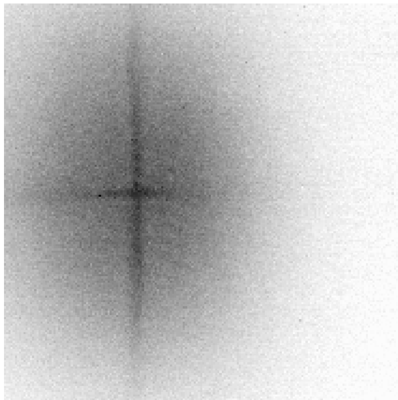
Spin changing coherent collision

$$2|0\rangle \leftrightarrow |+\rangle + |-\rangle$$

All-optical BEC

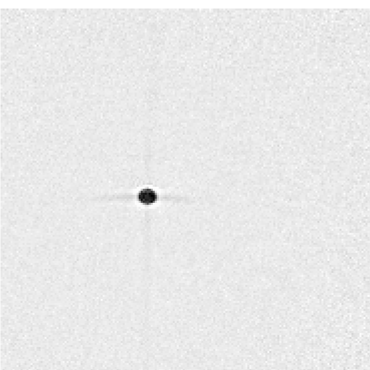
Loading from MOT

10^6 atoms loaded

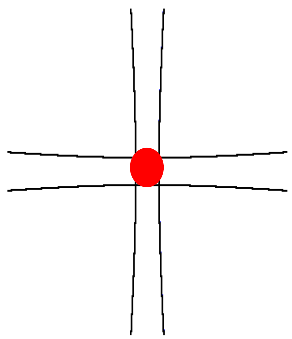


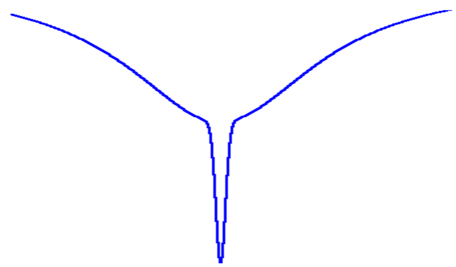
600 ms later

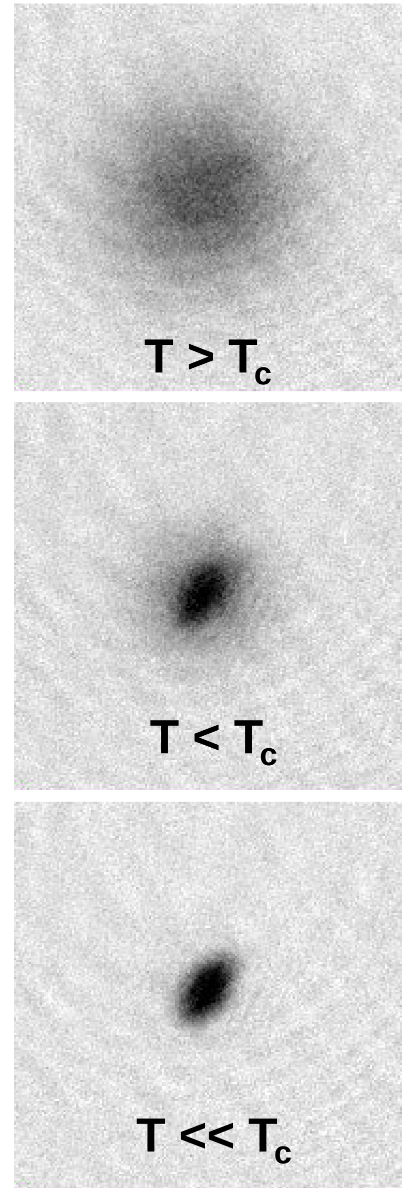
$n > 10^{14}$ cm^{-3}



- Two intersecting CO_2 laser beams
- Large loading volume provided by the 'wings'
- Tight confinement provided at the intersection





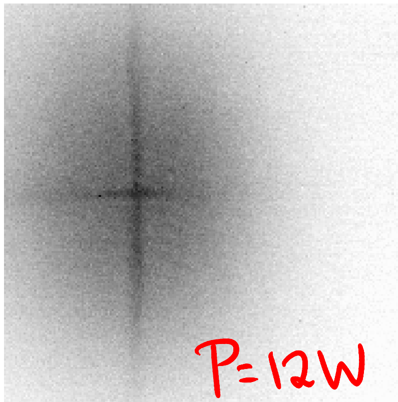


tight confinement → high density → fast evaporation

All-optical BEC

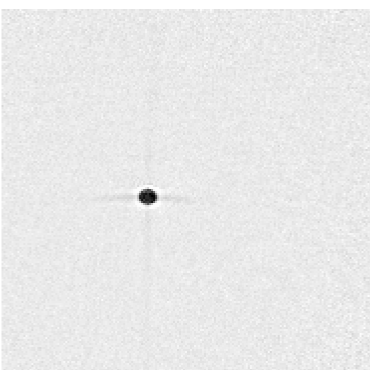
Loading from MOT

10^6 atoms loaded



600 ms later

$n > 10^{14}$ cm^{-3}



- Two intersecting CO_2 laser beams
- Large loading volume provided by the 'wings'
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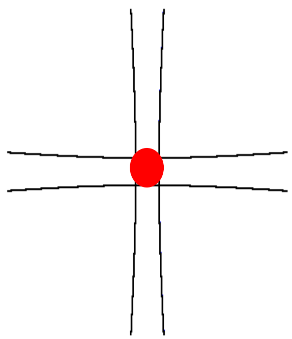
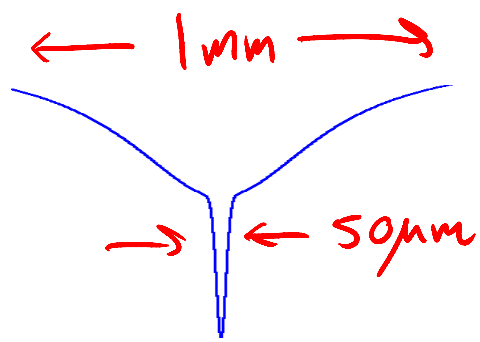
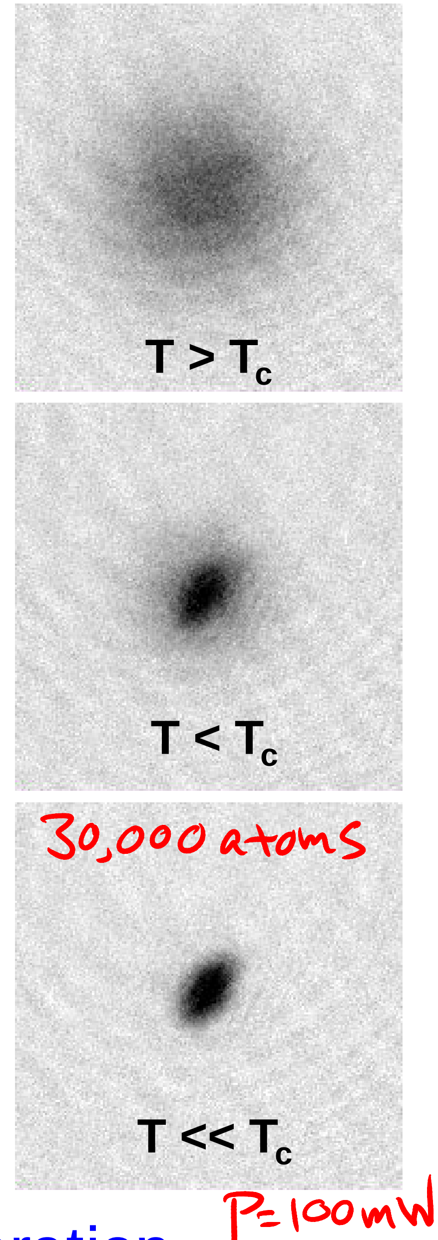
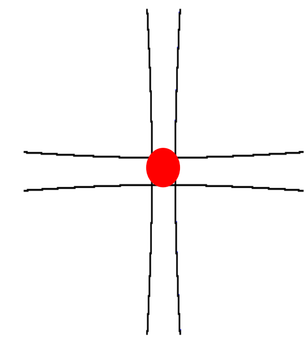




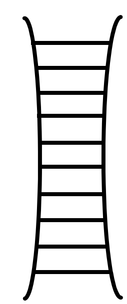


tight confinement → high density → fast evaporation

3 variations

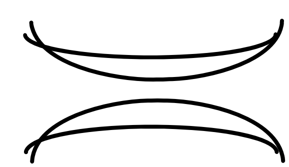


- **Cross trap**



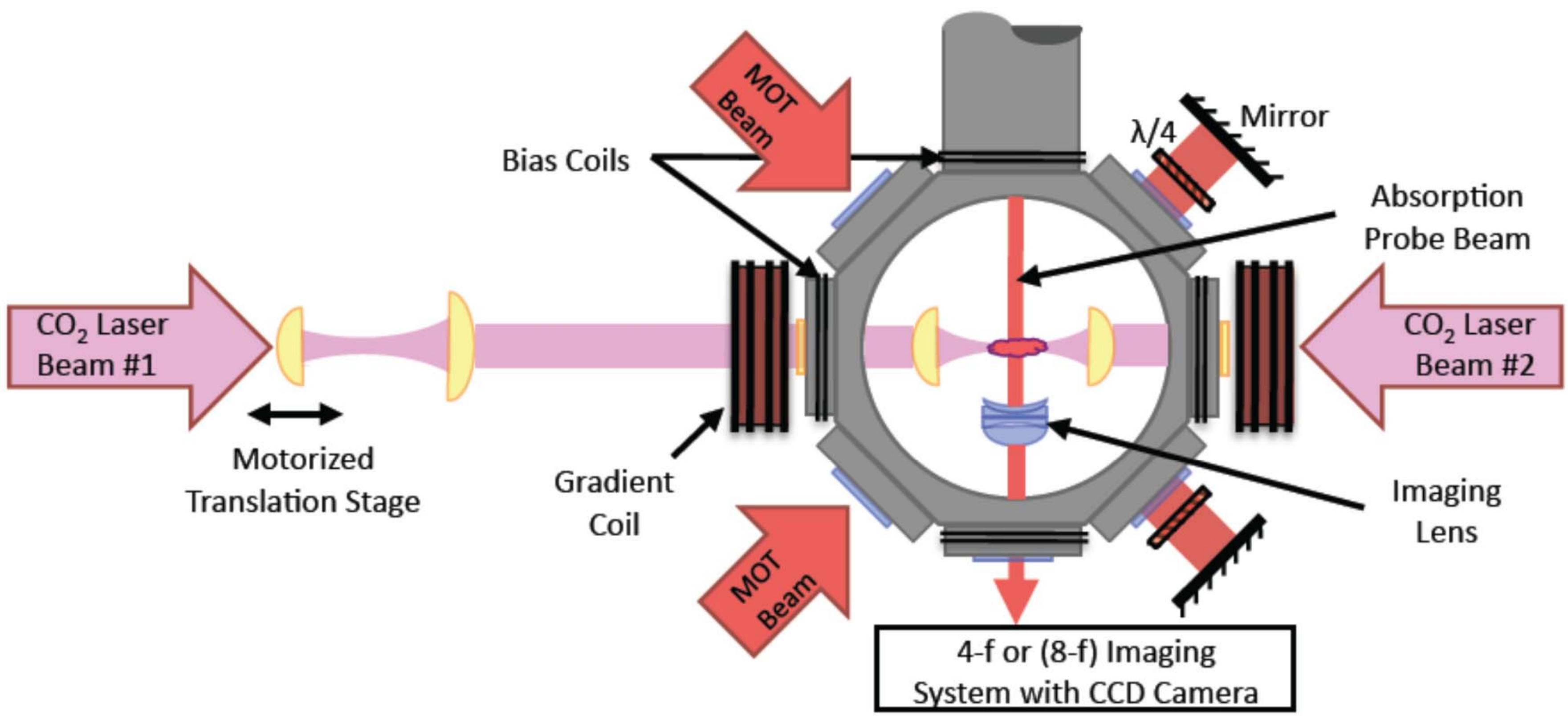
- **Large period (5 μm) 1-D lattice**

- **Single beam, variable focus trap**



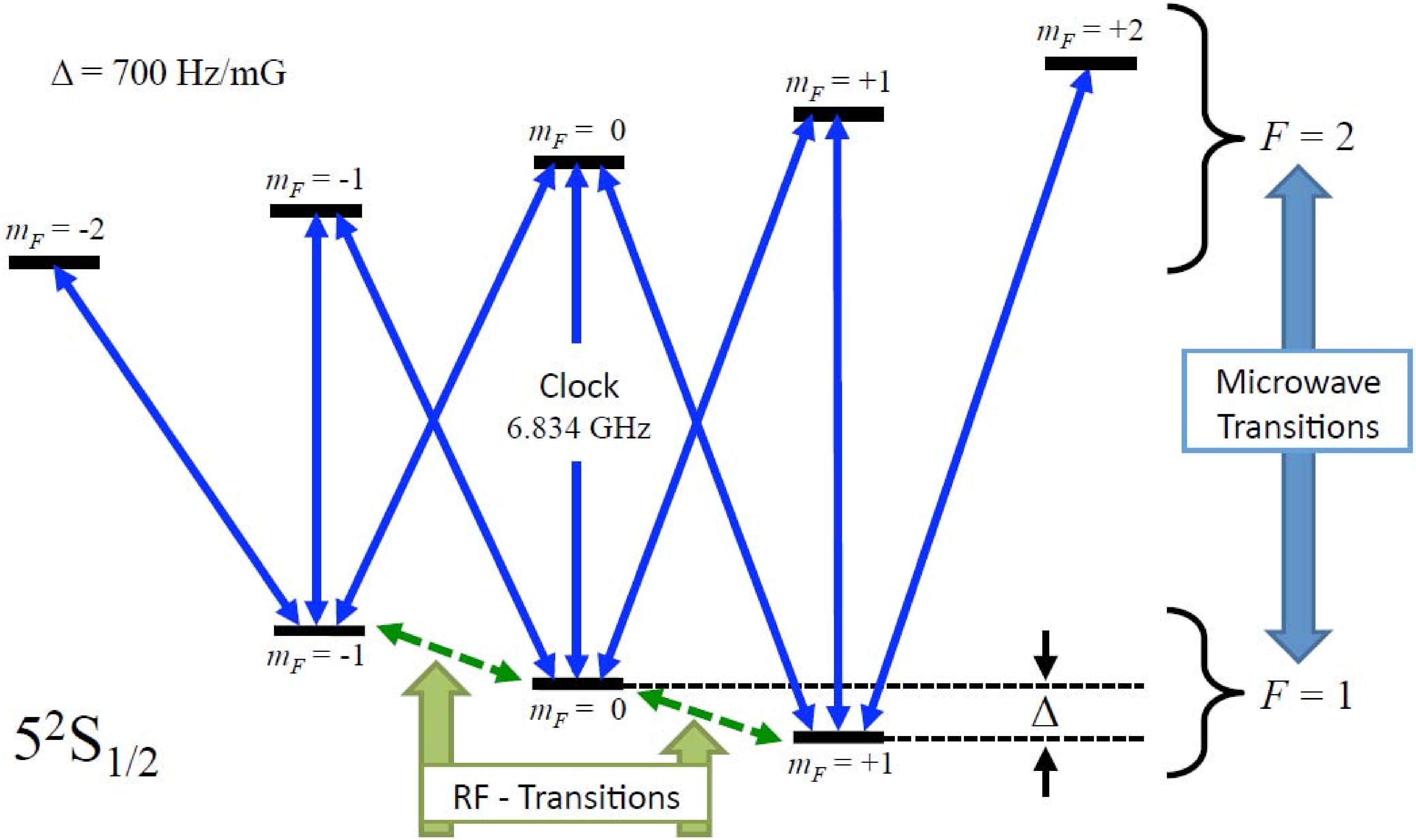
Common features:

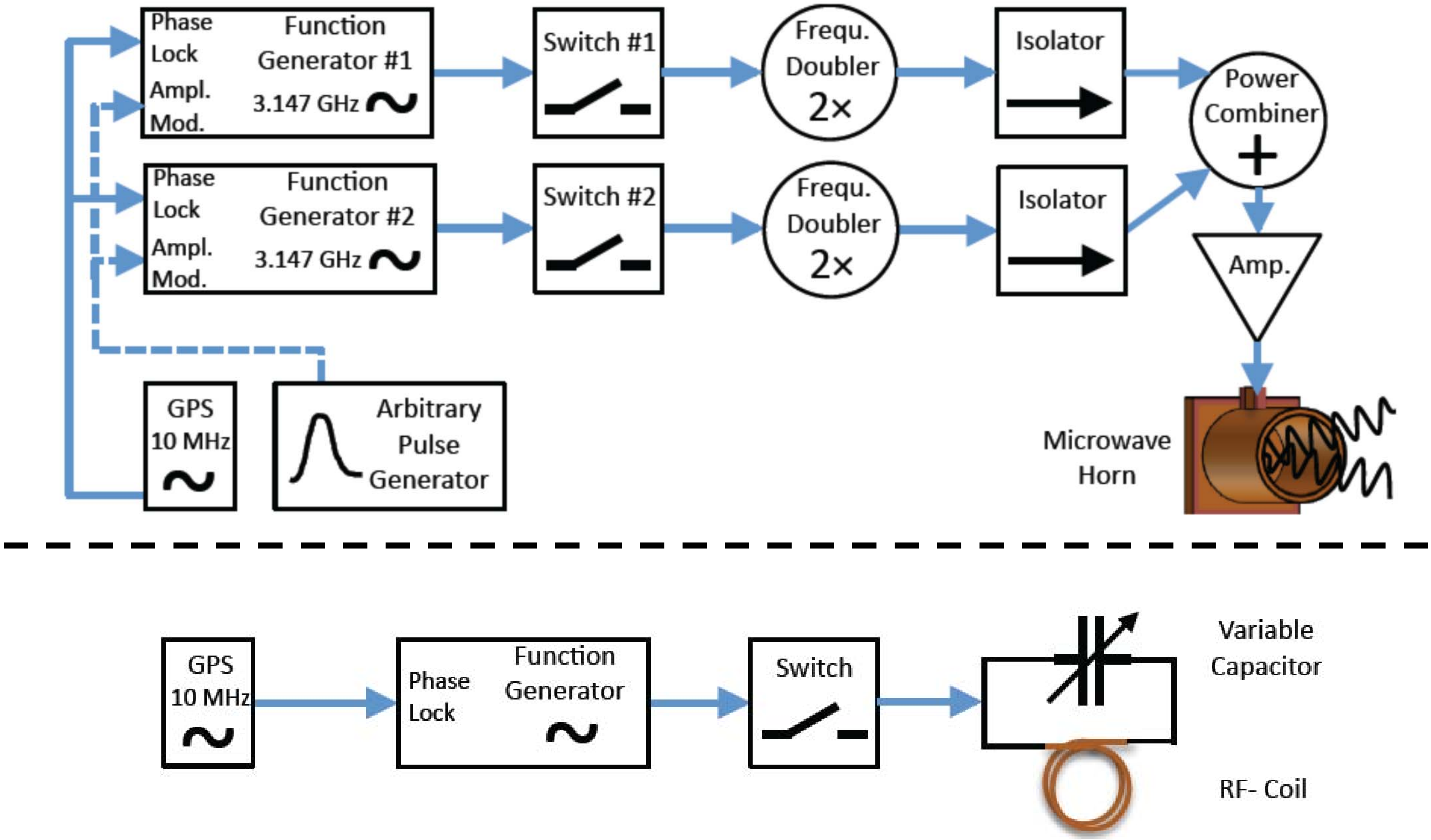
^{87}Rb
CO_2 laser
Simple MOT
< 2 s evaporation time



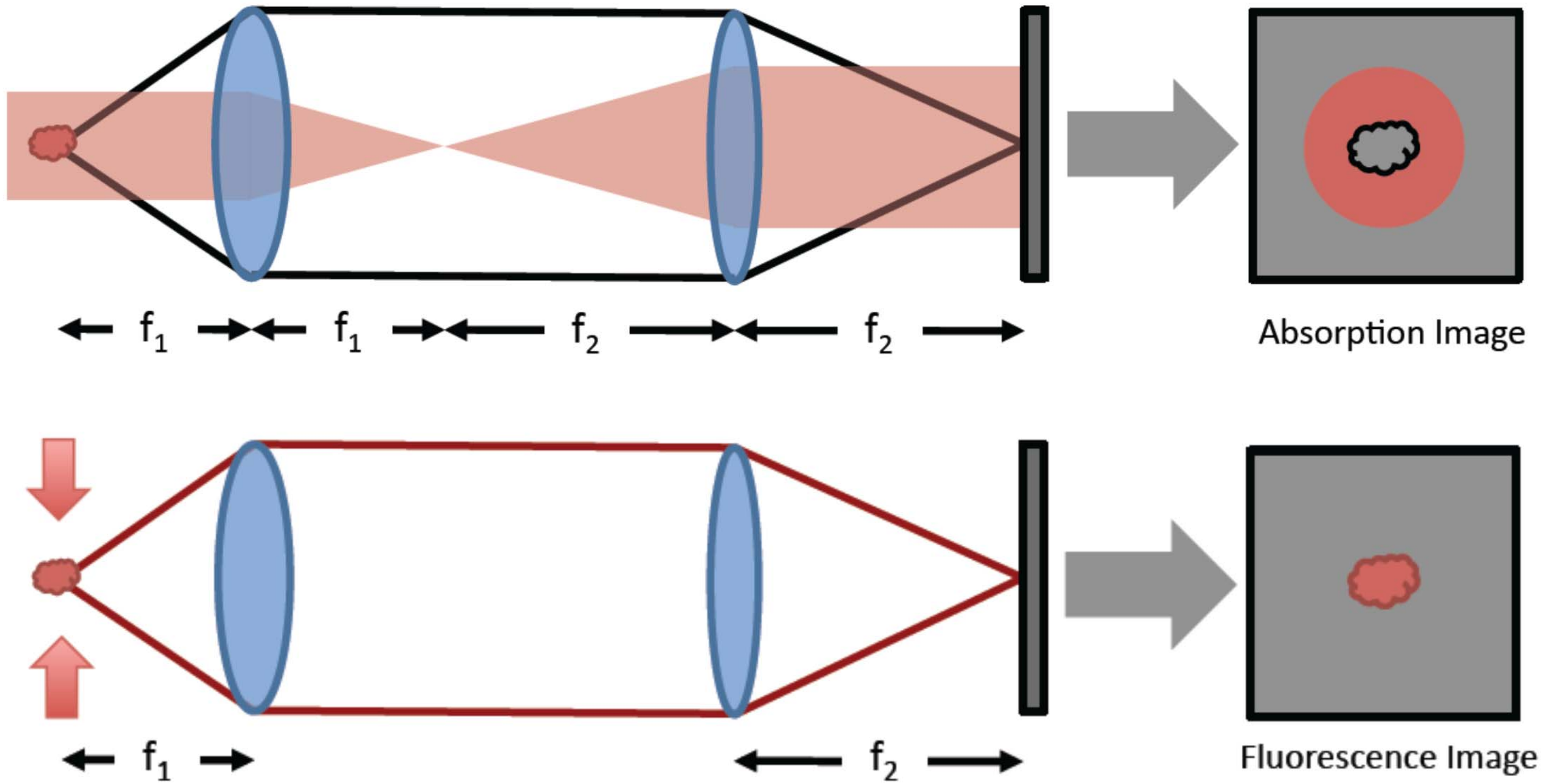
MOT Beam
Bias Coils
λ/4
Mirror
Absorption Probe Beam
CO2 Laser Beam #1
CO2 Laser Beam #2
Motorized Translation Stage
Gradient Coil
Imaging Lens
MOT Beam
4-f or (8-f) Imaging System with CCD Camera

Internal state manipulation




Phase Lock
Ampl. Mod.
Function Generator #1
3.147 GHz
Phase Lock
Ampl. Mod.
Function Generator #2
3.147 GHz
Switch #1
Switch #2
Frequ. Doubler 2×
Frequ. Doubler 2×
Isolator
Isolator
Power Combiner
Amp.
GPS 10 MHz
Arbitrary Pulse Generator
Microwave Horn
GPS 10 MHz
Phase Lock
Function Generator
Switch
Variable Capacitor
RF- Coil

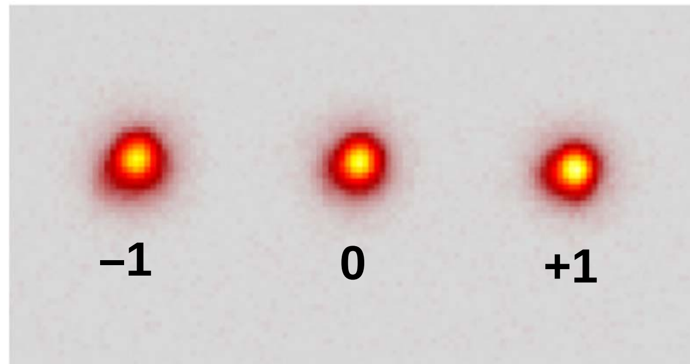
Imaging



Putting it together

- Load MOT
- Make BEC in B-field gradient
  - Pure $m_F = 0$ condensate
- Hold in high (2G) field
  - Spin does evolve
- Lower field
- Apply microwave and rf pulse to prepare initial state
- Allow spin dynamics to evolve
- Apply microwave and rf pulse to change measurement basis
- Release condensate from trap
- Let expand in Stern-Gerlach field to separate m_F states
- Image and count atoms

Measurement:



$$N = N_{-1} + N_0 + N_{+1}$$ **Total atom number**

$$\rho_0 = N_{+1}/N$$ **m_F = 0 population**

$$M = N_{+1} - N_{-1}$$ **Magnetization**

Hamiltonian -- Spin 1

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 r_i^2 + \sum_{i<j} V_{\sigma_i \sigma_j}(r_i - r_j)$$

Short-range interaction, transforms as scalar under spin rotations:

$$V_{\sigma_i \sigma_j}(r_i - r_j) = (c_0 + c_2 \mathbf{S_i} \cdot \mathbf{S_j}) \delta(r_i - r_j)$$

$|c_2| << c_0$

Spin interaction

Density interaction

Ho, PRL 81, 742 (1998)
Ohmni and Machida, 1998

^{87}Rb : $c_2 < 0$ **Ferromagnetic properties**

^{23}Na : $c_2 > 0$ **Anti-ferromagnetic properties**

Spinor condensates in optical traps

Multi-species BEC with rotational symmetry

$$\hat{H} = \int d^3 r \hat{\Psi}_i^+(\vec{r})[-\frac{\hbar^2\nabla^2}{2m}\delta_{ij} + U_{ij}(\vec{r})]\hat{\Psi}_j(\vec{r})$$

$$+\frac{c_0}{2}\int d^3 r \hat{\Psi}_i^+(\vec{r})\hat{\Psi}_j^+(\vec{r})\hat{\Psi}_j(\vec{r})\hat{\Psi}_i(\vec{r})$$

$$+\frac{c_2}{2}\int d^3 r \hat{\Psi}_i^+(\vec{r})\hat{\Psi}_j^+(\vec{r})(\hat{F}_m)_{ik}(\hat{F}_m)_{jl}\hat{\Psi}_j(\vec{r})\hat{\Psi}_i(\vec{r}),$$

$(i : -1, 0, 1)$

Ho, PRL98

Machida, JPS98

^{23}Na: F = 1, c_2 > 0, is anti-ferromagnetic

^{87}Rb: F = 1, c_2 < 0, is ferromagnetic

2^{nd} quantized picture

Hamiltonian

$$H = H_s + H_a \qquad\qquad H_s \gg H_a$$

$$H_s = \sum_\alpha \int d^3x \hat{\Psi}^\dagger_\alpha \left(\frac{\nabla^2}{2M} + V_T \right) \hat{\Psi}_\alpha + \frac{c_0}{2} \sum_{\alpha,\beta} \int \hat{\Psi}^\dagger_\alpha \hat{\Psi}^\dagger_\beta \hat{\Psi}_\alpha \hat{\Psi}_\beta d^3x$$

$$H_a = \frac{c_2}{2} \int \Big(\hat{\Psi}^\dagger_1 \hat{\Psi}^\dagger_1 \hat{\Psi}_1 \hat{\Psi}_1 + \hat{\Psi}^\dagger_{-1} \hat{\Psi}^\dagger_{-1} \hat{\Psi}_{-1} \hat{\Psi}_{-1} + 2\hat{\Psi}^\dagger_1 \hat{\Psi}^\dagger_0 \hat{\Psi}_1 \hat{\Psi}_0 + 2\hat{\Psi}^\dagger_{-1} \hat{\Psi}^\dagger_0 \hat{\Psi}_{-1} \hat{\Psi}_0 - 2\hat{\Psi}^\dagger_1 \hat{\Psi}^\dagger_{-1} \hat{\Psi}_1 \hat{\Psi}_{-1}$$
$$\underline{+2\hat{\Psi}^\dagger_1 \hat{\Psi}^\dagger_{-1} \hat{\Psi}_0 \hat{\Psi}_0 + 2\hat{\Psi}^\dagger_0 \hat{\Psi}^\dagger_0 \hat{\Psi}_1 \hat{\Psi}_{-1} \Big) d^3x}$$

$$2\left|0\right\rangle \rightleftarrows \left|+\right\rangle + \left|-\right\rangle$$

Mean-field Coupled Gross-Pitaevskii Eqn.

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 r_i^2 + \sum_{i<j} (c_0 + c_2 \vec{S}_i \cdot \vec{S}_j) \delta(r_i - r_j)$$

$$\vec{\phi}(\vec{r}) = \begin{pmatrix} \phi_1 & \phi_0 & \phi_{-1} \end{pmatrix}^T$$ **Condensate wave function**

$$i\hbar \frac{\partial \phi_1}{\partial t} = L_1 \phi_1 + c_0 n \phi_1 + c_2 (n_1 + n_0 - n_{-1}) \phi_1 + c_2 \phi_{-1}^* \phi_0^2$$

$$i\hbar \frac{\partial \phi_0}{\partial t} = L_0 \phi_0 + c_0 n \phi_0 + c_2 (n_1 + n_{-1}) \phi_0 + 2 c_2 \phi_0^* \phi_1 \phi_{-1}$$

$$i\hbar \frac{\partial \phi_{-1}}{\partial t} = L_{-1} \phi_{-1} + c_0 n \phi_{-1} + c_2 (-n_1 + n_0 + n_{-1}) \phi_{-1} + c_2 \phi_1^* \phi_0^2$$

$L_{\pm 1,0} = (-\hbar^2 \nabla^2 / 2m + U_{\pm 1,0} - \mu)$ and $n = n_1 + n_0 + n_{-1}$.

Cross-phase modulation

Coherent spin (4-wave) mixing

Single mode approximation (SMA)

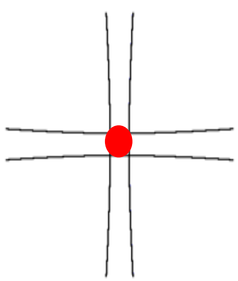
For condensates with spatial size smaller than spin healing length, the spin components have same spatial wavefunction

$$\vec{\phi}(\vec{r}) = \begin{pmatrix} \phi_1(\vec{r}) \\ \phi_0(\vec{r}) \\ \phi_{-1}(\vec{r}) \end{pmatrix} \rightarrow \phi(\vec{r}) \begin{pmatrix} \sqrt{\rho_1}\, e^{i\theta_1} \\ \sqrt{\rho_0}\, e^{i\theta_0} \\ \sqrt{\rho_{-1}}\, e^{i\theta_{-1}} \end{pmatrix}$$

Results in considerable simplification

Spin healing length: $\xi_s = \dfrac{h}{\sqrt{2mc_2 n}} \sim 10-15\ \mu\mathrm{m}$

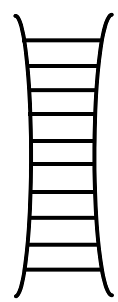
Cross trap



~ spherical

$(2\rho_c, 2z_c) \sim (7, 7)\ \mu$m

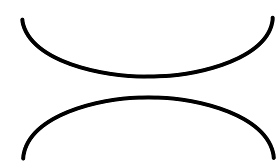
1-D lattice



disk

$(2\rho_c, 2z_c) \sim (1, 10)\ \mu$m

Single focus



cigar

$(2\rho_c, 2z_c) \sim (7, 70)\ \mu$m

Quantum solution

Spin part of Hamiltonian

$$H_a = \frac{c_2}{2}\int\Big(\hat\Psi_1^\dagger\hat\Psi_1^\dagger\hat\Psi_1\hat\Psi_1 + \hat\Psi_{-1}^\dagger\hat\Psi_{-1}^\dagger\hat\Psi_{-1}\hat\Psi_{-1} + 2\hat\Psi_1^\dagger\hat\Psi_0^\dagger\hat\Psi_1\hat\Psi_0 + 2\hat\Psi_{-1}^\dagger\hat\Psi_0^\dagger\hat\Psi_{-1}\hat\Psi_0 - 2\hat\Psi_1^\dagger\hat\Psi_{-1}^\dagger\hat\Psi_1\hat\Psi_{-1}$$
$$+2\hat\Psi_1^\dagger\hat\Psi_{-1}^\dagger\hat\Psi_0\hat\Psi_0 + 2\hat\Psi_0^\dagger\hat\Psi_0^\dagger\hat\Psi_1\hat\Psi_{-1}\Big)d^3x$$

Single spatial mode approximation $\longrightarrow$ $\hat\Psi_\kappa \approx \hat a_\kappa\,\phi(\mathbf{r})$

$$H_a = \lambda_a'(a_1^\dagger a_1^\dagger a_1 a_1 + a_{-1}^\dagger a_{-1}^\dagger a_{-1}a_{-1} - 2a_1^\dagger a_{-1}^\dagger a_1 a_{-1} + 2a_1^\dagger a_0^\dagger a_1 a_0 + 2a_{-1}^\dagger a_0^\dagger a_{-1}a_0$$
$$+2a_0^\dagger a_0^\dagger a_1 a_{-1} + 2a_1^\dagger a_{-1}^\dagger a_0 a_0)$$

$$2\lambda_i' = \lambda_i\int|\phi(\mathbf{r})^4|d^3r$$

Introduce angular momentum operators

$$S_- \equiv \sqrt{2}\left(a_1^\dagger a_0 + a_0^\dagger a_{-1}\right) \quad S_+ \equiv \sqrt{2}\left(a_0^\dagger a_1 + a_{-1}^\dagger a_0\right) \quad S_z \equiv \sqrt{2}\left(a_1^\dagger a_1 - a_{-1}^\dagger a_{-1}\right)$$

$$\boxed{H_a = \lambda_a' S^2}$$

Step 2

Single spatial mode approximation $\longrightarrow$ $\hat{\Psi}_\kappa \approx \hat{a}_\kappa \phi(\mathbf{r})$

$$H_a = \lambda_a'(a_1^\dagger a_1^\dagger a_1 a_1 + a_{-1}^\dagger a_{-1}^\dagger a_{-1} a_{-1} - 2a_1^\dagger a_{-1}^\dagger a_1 a_{-1} + 2a_1^\dagger a_0^\dagger a_1 a_0 + 2a_{-1}^\dagger a_0^\dagger a_{-1} a_0$$
$$+2a_0^\dagger a_0^\dagger a_1 a_{-1} + 2a_1^\dagger a_{-1}^\dagger a_0 a_0)$$

Derive Heisenberg equations of motion

$$i\hbar\frac{\partial \hat{a}_1}{\partial t} = 2\lambda_a'\left(\hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 - \hat{a}_{-1}^\dagger \hat{a}_1 \hat{a}_{-1} + \hat{a}_0^\dagger \hat{a}_1 \hat{a}_0 + \hat{a}_{-1}^\dagger \hat{a}_0 \hat{a}_0\right)$$
$$i\hbar\frac{\partial \hat{a}_0}{\partial t} = 2\lambda_a'\left(\hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 + \hat{a}_{-1}^\dagger \hat{a}_0 \hat{a}_{-1} + 2\hat{a}_0^\dagger \hat{a}_1 \hat{a}_{-1}\right)$$
$$i\hbar\frac{\partial \hat{a}_{-1}}{\partial t} = 2\lambda_a'\left(\hat{a}_{-1}^\dagger \hat{a}_{-1} \hat{a}_{-1} - \hat{a}_1^\dagger \hat{a}_1 \hat{a}_{-1} + \hat{a}_0^\dagger \hat{a}_0 \hat{a}_{-1} + \hat{a}_1^\dagger \hat{a}_0 \hat{a}_0\right)$$

Step 3

Heisenberg equations of motion

$$i\hbar\frac{\partial \hat{a}_1}{\partial t} = 2\lambda'_a\left(\hat{a}_1^\dagger\hat{a}_1\hat{a}_1 - \hat{a}_{-1}^\dagger\hat{a}_1\hat{a}_{-1} + \hat{a}_0^\dagger\hat{a}_1\hat{a}_0 + \hat{a}_{-1}^\dagger\hat{a}_0\hat{a}_0\right)$$

$$i\hbar\frac{\partial \hat{a}_0}{\partial t} = 2\lambda'_a\left(\hat{a}_1^\dagger\hat{a}_1\hat{a}_0 + \hat{a}_{-1}^\dagger\hat{a}_0\hat{a}_{-1} + 2\hat{a}_0^\dagger\hat{a}_1\hat{a}_{-1}\right)$$

$$i\hbar\frac{\partial \hat{a}_{-1}}{\partial t} = 2\lambda'_a\left(\hat{a}_{-1}^\dagger\hat{a}_{-1}\hat{a}_{-1} - \hat{a}_1^\dagger\hat{a}_1\hat{a}_{-1} + \hat{a}_0^\dagger\hat{a}_0\hat{a}_{-1} + \hat{a}_1^\dagger\hat{a}_0\hat{a}_0\right)$$

Make mean-field approximation: $\hat{a}_\kappa \to \sqrt{N}\zeta_\kappa,\ \hat{a}_\kappa^\dagger \to \sqrt{N}\zeta_\kappa^*$

$$i\hbar\dot{\zeta}_1 = c[(\rho_1 + \rho_0 - \rho_{-1})\zeta_1 + \zeta_0^2\zeta_{-1}^*]$$

$$i\hbar\dot{\zeta}_0 = c[(\rho_1 + \rho_{-1})\zeta_0 + 2\zeta_1\zeta_{-1}\zeta_0^*]$$

$$i\hbar\dot{\zeta}_{-1} = c[(\rho_{-1} + \rho_0 - \rho_1)\zeta_{-1} + \zeta_0^2\zeta_1^*]$$

$c = 2\lambda'_a N$ and $\rho_i = |\zeta_i|^2 = N_i/N$

Step 4

Mean-field approximation

$$
\begin{aligned}
i\hbar\dot{\zeta}_1 &= c[(\rho_1+\rho_0-\rho_{-1})\zeta_1+\zeta_0^2\zeta_{-1}^*] \\
i\hbar\dot{\zeta}_0 &= c[(\rho_1+\rho_{-1})\zeta_0+2\zeta_1\zeta_{-1}\zeta_0^*] \\
i\hbar\dot{\zeta}_{-1} &= c[(\rho_{-1}+\rho_0-\rho_1)\zeta_{-1}+\zeta_0^2\zeta_1^*]
\end{aligned}
$$

$$
\begin{aligned}
\zeta_1 &= \sqrt{\frac{1-\rho_0+m}{2}}e^{i\frac{\theta_s+\theta_m}{2}} \\
\zeta_0 &= \sqrt{\rho_0} \\
\zeta_{-1} &= \sqrt{\frac{1-\rho_0-m}{2}}e^{i\frac{\theta_s-\theta_m}{2}}
\end{aligned}
$$

$$\theta_m = \theta_1 - \theta_{-1}$$

$$\theta_s = \theta_1 + \theta_{-1} - 2\theta_0$$

$$m = (N_1 - N_{-1})/N = \rho_1 - \rho_{-1}$$

“Pendulum” equations of motion

$$
\begin{aligned}
\dot{\rho}_0 &= \frac{2c}{\hbar}\rho_0\sqrt{(1-\rho_0)^2-m^2}\sin\theta_s \\
\dot{\theta}_s &= \frac{2c}{\hbar}\left[(1-2\rho_0)+\frac{(1-\rho_0)(1-2\rho_0)-m^2}{\sqrt{(1-\rho_0)^2-m^2}}\cos\theta_s\right]
\end{aligned}
$$

Step 4

"Pendulum" equations of motion

$$\theta_m = \theta_1 - \theta_{-1}$$

$$\theta_s = \theta_1 + \theta_{-1} - 2\theta_0$$

$$m = (N_1 - N_{-1})/N = \rho_1 - \rho_{-1}$$

$$\dot{\rho}_0 = \frac{2c}{\hbar}\rho_0\sqrt{(1-\rho_0)^2 - m^2}\sin\theta_s$$

$$\dot{\theta}_s = \frac{2c}{\hbar}\left[(1-2\rho_0) + \frac{(1-\rho_0)(1-2\rho_0) - m^2}{\sqrt{(1-\rho_0)^2 - m^2}}\cos\theta_s\right]$$

Classical Hamiltonian for system

$$\mathcal{E} = \frac{c}{2}m^2 + c\rho_0\left[(1-\rho_0) + \sqrt{(1-\rho_0)^2 - m^2}\cos\theta_s\right]$$

Single mode approximation (SMA)

Spin energy functional:

$$E = c\rho_0[(1-\rho_0 + \sqrt{(1-\rho_0)^2 - M^2}\cos\theta)]$$

Just two dynamical variables:

$\rho_0(t)$ **: fractional population of the m_F = 0 state**

$\theta(t) \equiv \theta_+ + \theta_- - 2\theta_0$ **relative phase of the spinor components**

Fixed parameters:

$$c = c_2 \int |\phi(r)|^4 d^3r$$

Spin-dependent interaction strength

$$M = \rho_1 - \rho_{-1} = \langle L_z \rangle$$

'Magnetization' is constant of motion:

$2|0\rangle \rightarrow |+\rangle + |-\rangle$ **conserves *M***

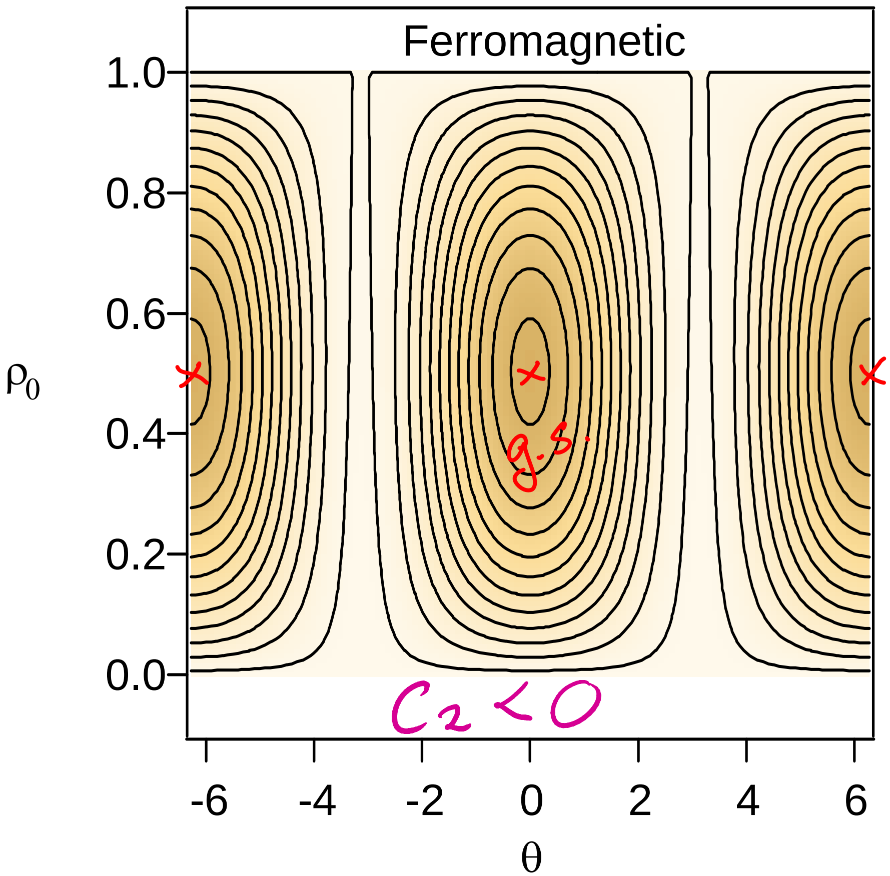
Population of other components

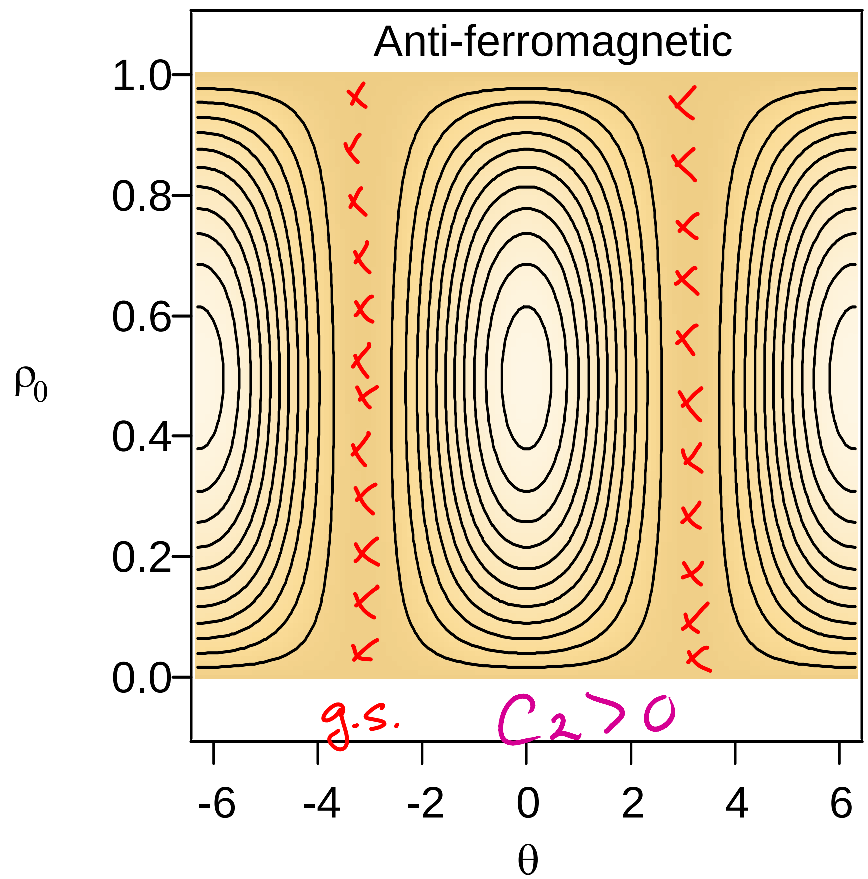
$$\rho_{\pm 1} = (1 - \rho_0 \pm M)/2$$

"Mean field ground state of a spin-1 condensate in a magnetic field,"
Wenxian Zhang, Su Yi and Li You, 2003

Spinor energy contours

$M=0,\ B=0.$



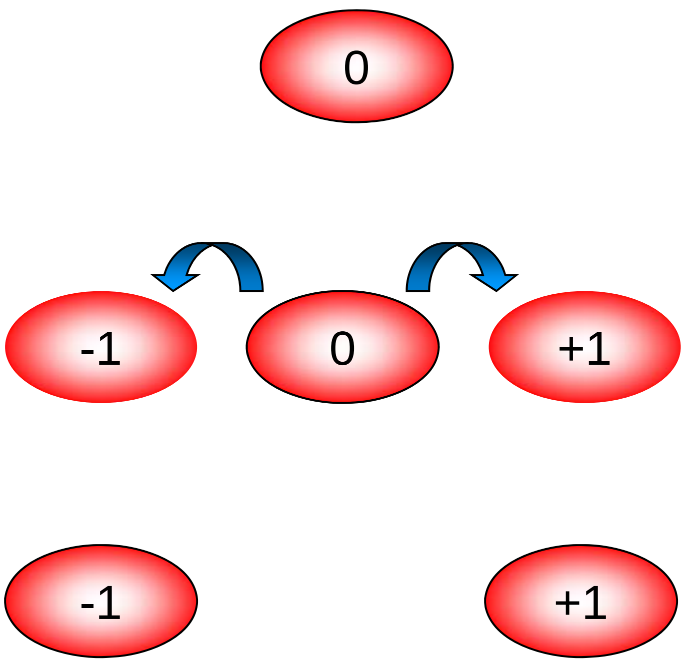


$$E = c\rho_0[(1-\rho_0+\sqrt{(1-\rho_0)^2 - M^2}\cos\theta)]$$

Zeeman energies

$$2|0\rangle \rightarrow |+\rangle + |-\rangle$$

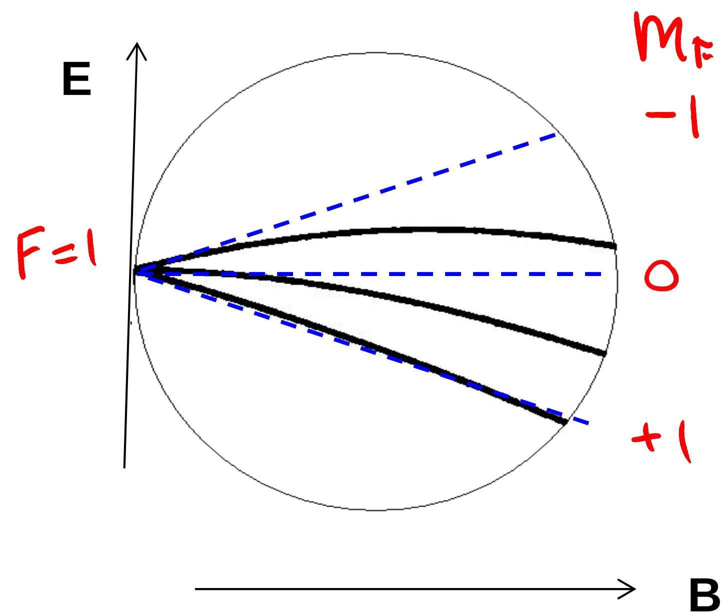
(Huge) zeeman energies cancel to lowest (linear) order



spin mixing

Quadratic Zeeman energy favors $m_F = 0$

$$q = (E_{+1} + E_{-1} - 2E_0)/2 \approx 72\ \text{Hz/gauss}^2$$

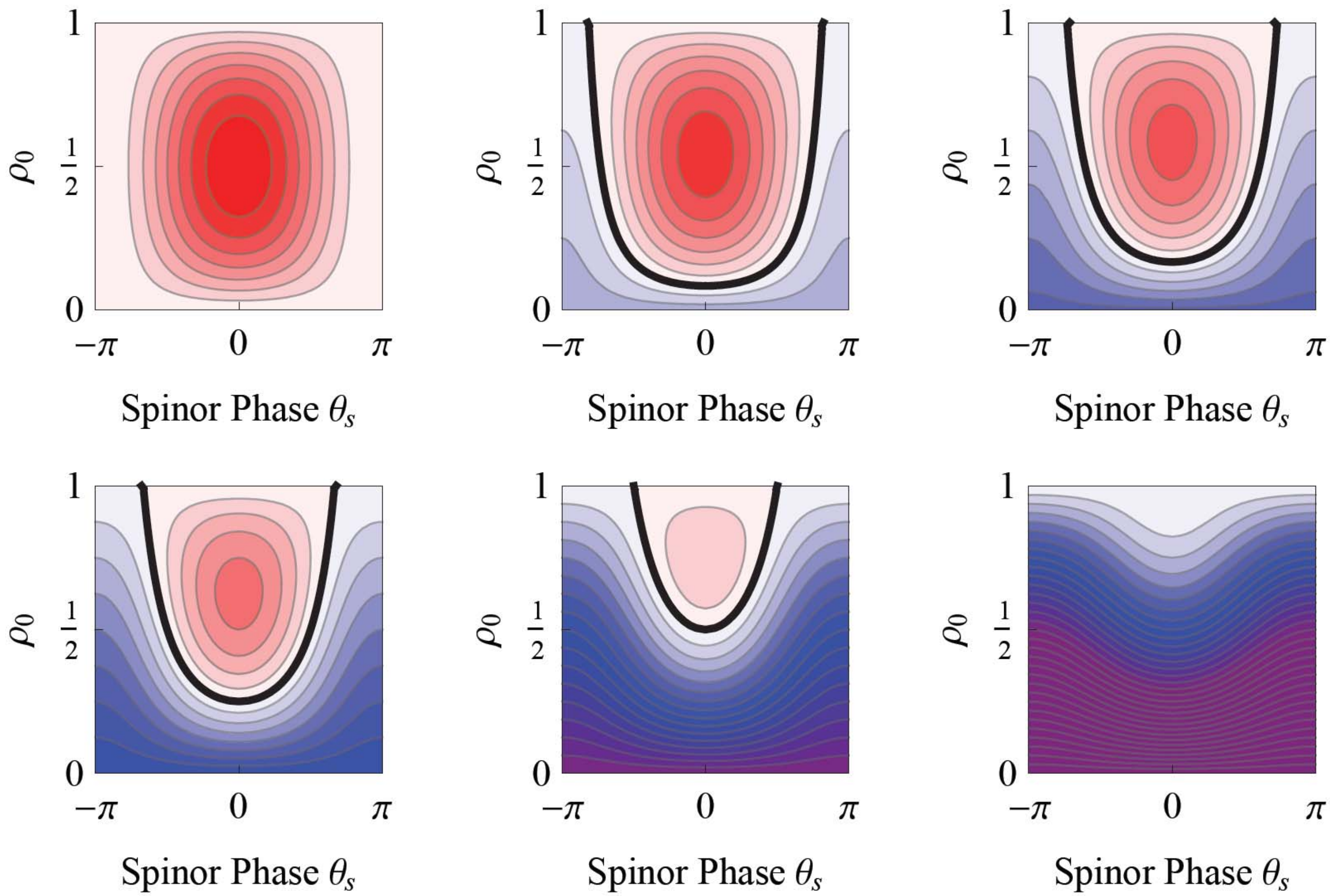


Spinor energy contours

$c < 0$

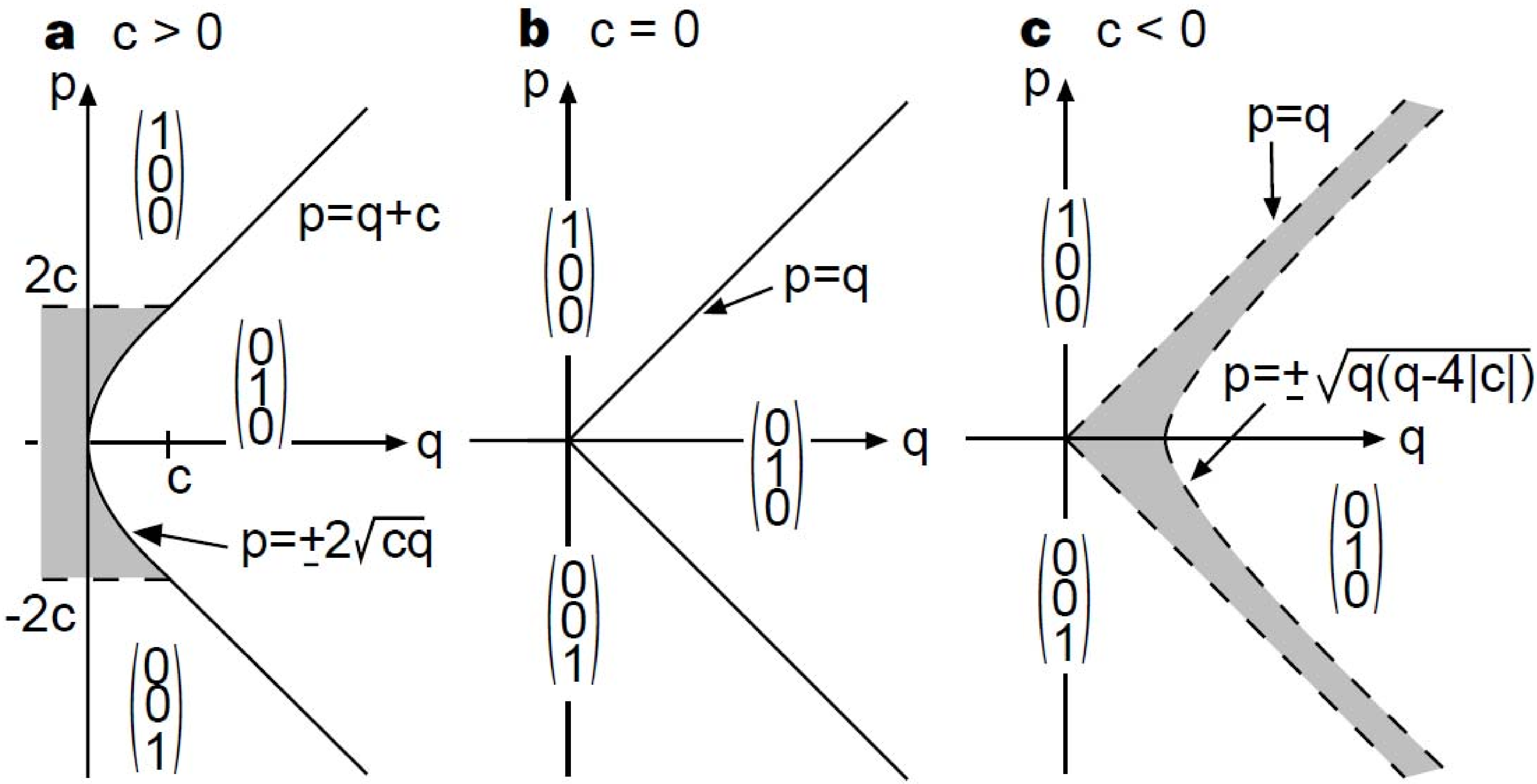
$q > 0$

$M = 0$

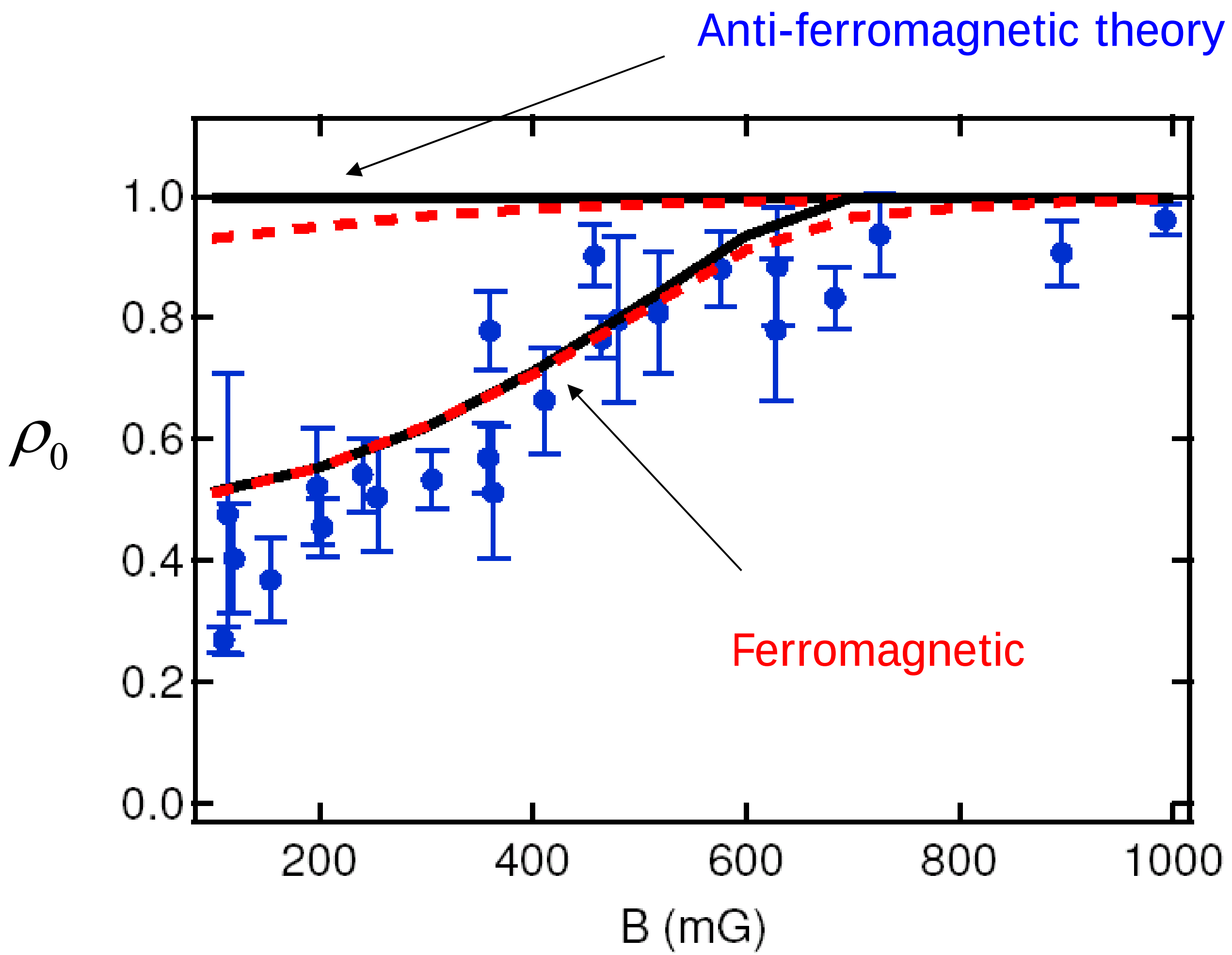


$$E = c\rho_0[(1-\rho_0 + \sqrt{(1-\rho_0)^2 - M^2}\cos\theta)] + q(1-\rho_0),$$

Phase diagram



Ferromagnetic Spinor ground state vs magnetic field



^{87}Rb F = 1

Ferromagnetic behavior

"Observation of spinor dynamics in optically trapped ^{87}Rb Bose-Einstein Condensates," M.-S. Chang, C. D. Hamley, M. D. Barrett,_ J. A. Sauer, K. M. Fortier, W. Zhang, L. You, and M. S. Chapman, Phys. Rev. Lett, 92, 140403 (2004).

Spinor experiments at Georgia Tech

Mean-field regime

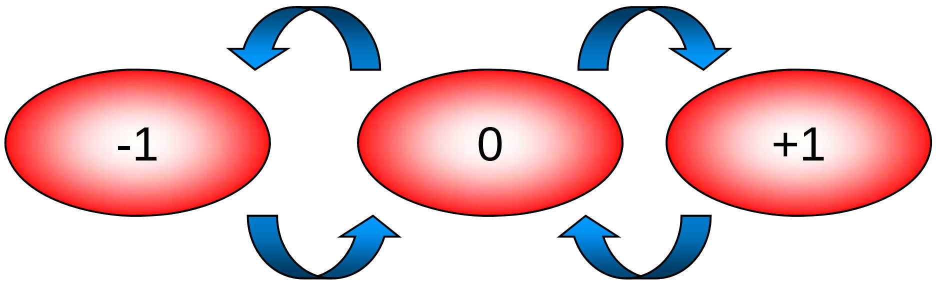
- Observation of F = 1, ^{87}Rb spinor in first all-optical BEC
- Determination of the phase diagram and sign of c_2 (ferromagnetic)
  Measurements of ground state vs B field
- Coherent spin dynamics at low B field
  -- spin oscillations & direct measurement of c_2
- Coherent spin dynamics at high B field
  -- internal AC Josephson effect
- Coherent control of spinor dynamics
  -- determination of $c_2 < 0$ via relaxation to ground state

Quantum regime

- Spin-Nematic squeezing in a spin-1 condensate
- Non-equilibrium quantum spin dynamics

Coherent spin mixing in a F = 1 ferromagnetic condensate

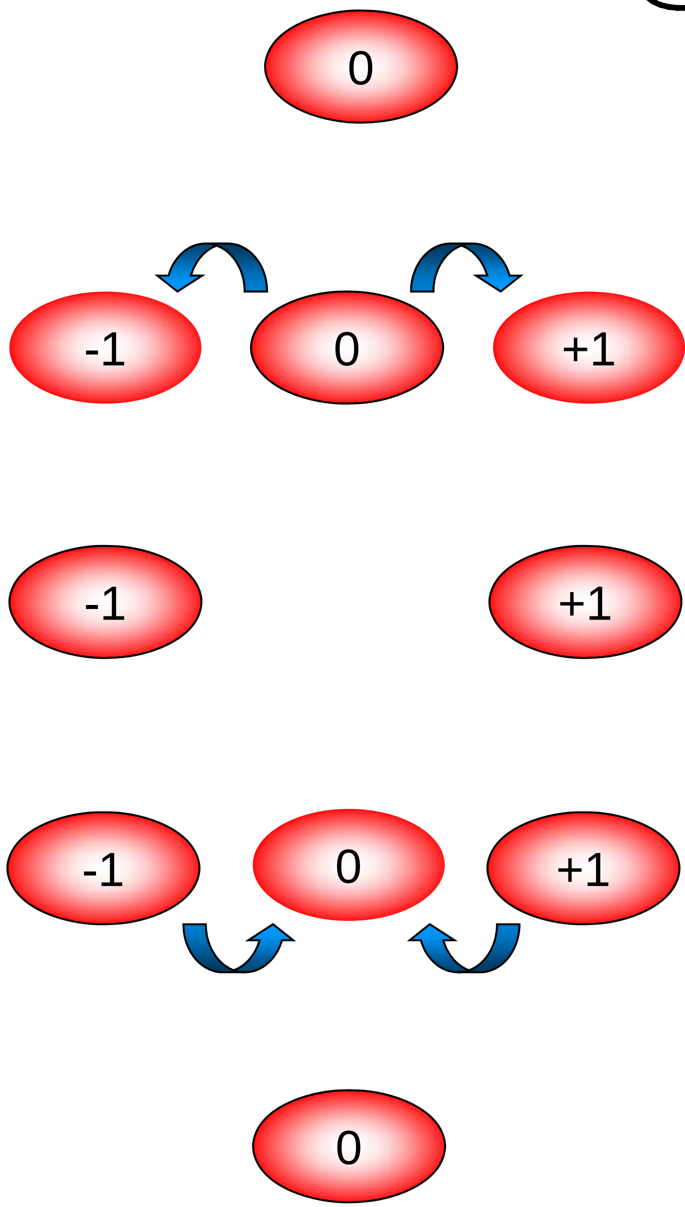
$$2|0\rangle \rightarrow |+\rangle + |-\rangle$$



$$|+\rangle + |-\rangle \rightarrow 2|0\rangle$$

Josephson dynamics driven only by spin-dependent interactions
A new macroscopic quantum system

Coherent spin mixing



$$2|0\rangle \to |+\rangle + |-\rangle$$

$$|+\rangle + |-\rangle \to 2|0\rangle$$

Meta-stable spin configuration

At t=0: (n_1, n_0, n_{-1}) = (0, 1, 0)

$$i\hbar\frac{\partial\phi_1}{\partial t} = L_1\phi_1 + c_0 n\phi_1 + c_2(n_1 + n_0 - n_{-1})\phi_1 + c_2\phi_{-1}^*\phi_0^2$$

$$i\hbar\frac{\partial\phi_0}{\partial t} = L_0\phi_0 + c_0 n\phi_0 + c_2(n_1 + n_{-1})\phi_0 + 2c_2\phi_0^*\phi_1\phi_{-1}$$

$$i\hbar\frac{\partial\phi_{-1}}{\partial t} = L_{-1}\phi_{-1} + c_0 n\phi_{-1} + c_2(-n_1 + n_0 + n_{-1})\phi_{-1} + c_2\phi_1^*\phi_0^2$$

where $L_{\pm 1,0} = (-\frac{\hbar^2\nabla^2}{2m} + U_{\pm 1,0} - \mu)$ and $n = n_1 + n_0 + n_{-1}$.

Spin mixing is noise driven.

GaTech (2002)

Meta-stable spin configuration

At t=0: $(n_1, n_0, n_{-1}) = (0, 1, 0)$

$$i\hbar \frac{\partial \phi_1}{\partial t} = L_1 \cancel{\phi_1} + c_0 n \cancel{\phi_1} + \cancel{c_2 (n_1 + n_0 - n_{-1}) \phi_1} + c_2 \cancel{\phi_{-1}^*} \phi_0^2$$

$$i\hbar \frac{\partial \phi_0}{\partial t} = L_0 \phi_0 + c_0 n \phi_0 + c_2 (\cancel{n_1} + \cancel{n_{-1}}) \phi_0 + 2 c_2 \phi_0^* \cancel{\phi_1} \phi_{-1}$$

$$i\hbar \frac{\partial \phi_{-1}}{\partial t} = L_{-1} \cancel{\phi_{-1}} + c_0 n \cancel{\phi_{-1}} + \cancel{c_2 (-n_1 + n_0 + n_{-1}) \phi_{-1}} + c_2 \cancel{\phi_1^*} \phi_0^2$$

where $L_{\pm 1,0} = (-\frac{\hbar^2 \nabla^2}{2m} + U_{\pm 1,0} - \mu)$ and $n = n_1 + n_0 + n_{-1}$.

Spin mixing is noise driven.

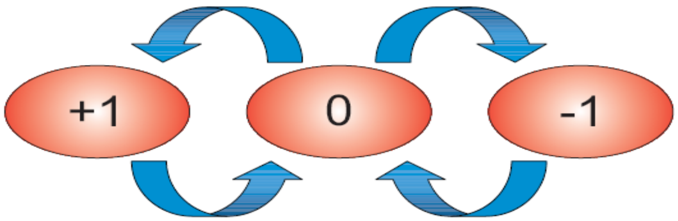
GaTech (2002)

Observing coherent spin dynamics

(First) Strategy

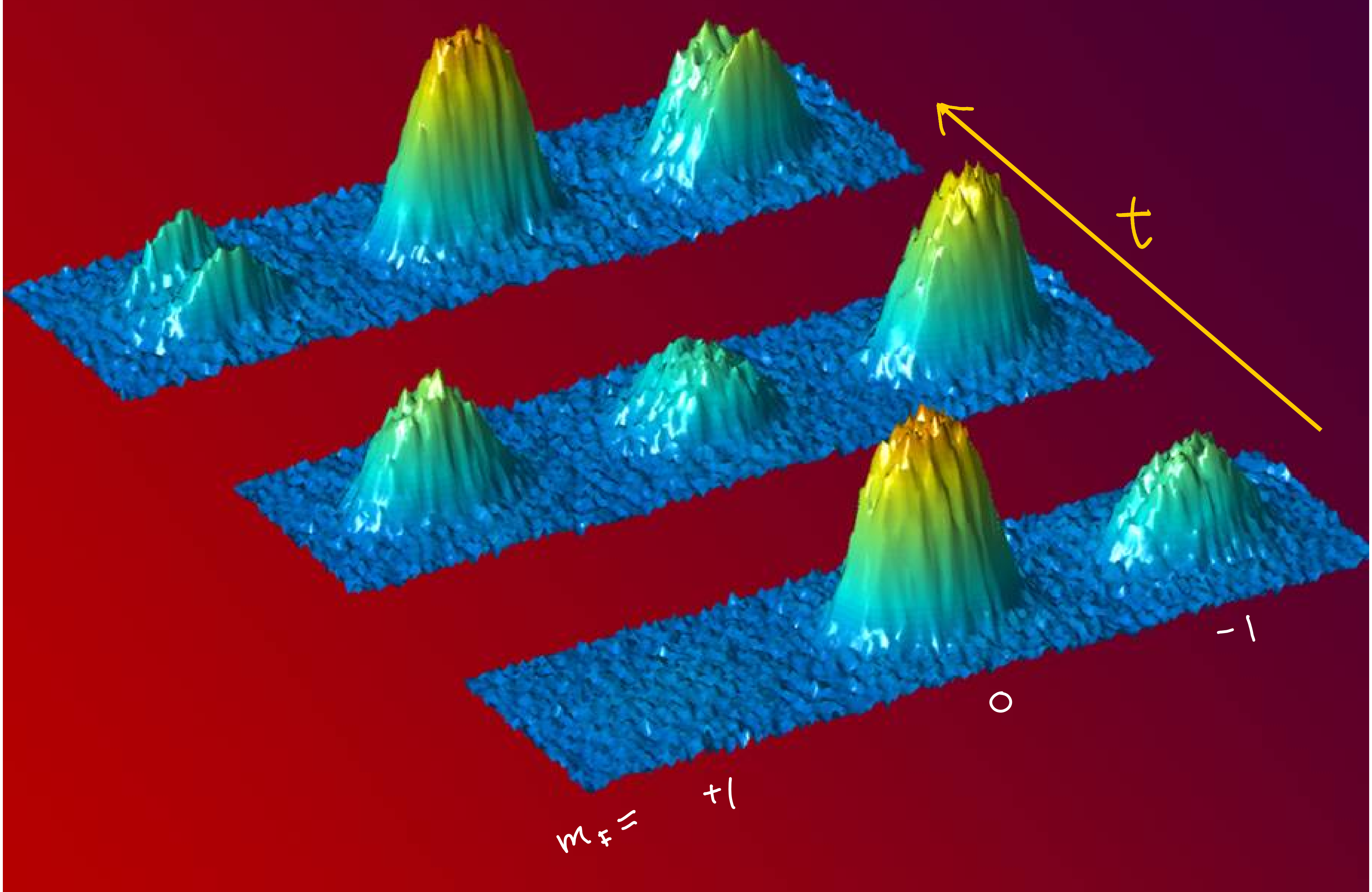
- Start with a state that is not metastable
- Use condensate satisfying SMA
- Use phase coherent state preparation
- Cancel magnetic field gradients

$$2|0\rangle \rightarrow |+\rangle + |-\rangle$$

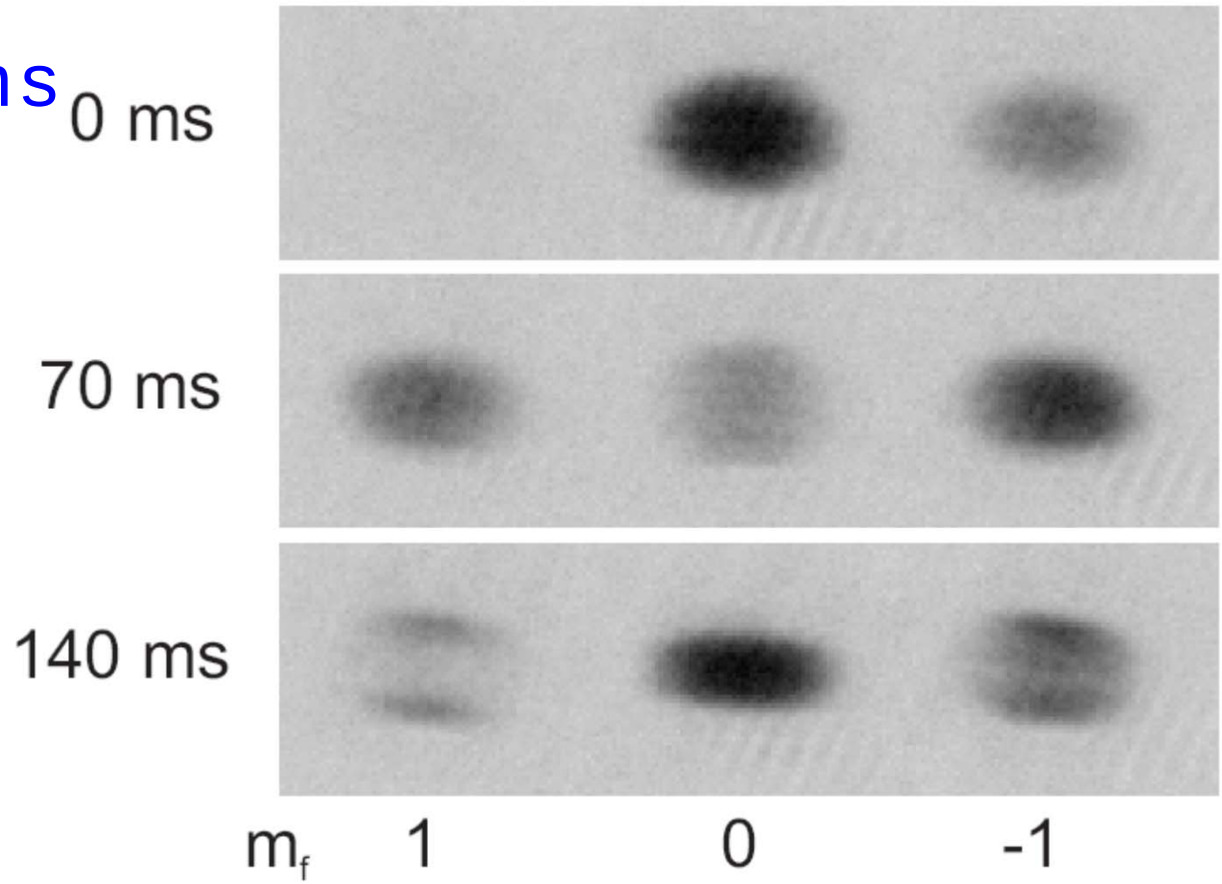


$$|+\rangle + |-\rangle \rightarrow 2|0\rangle$$

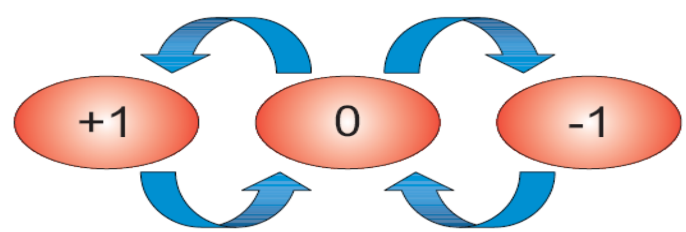
Coherent dynamics in a spin-1 condensate



Spin oscillations

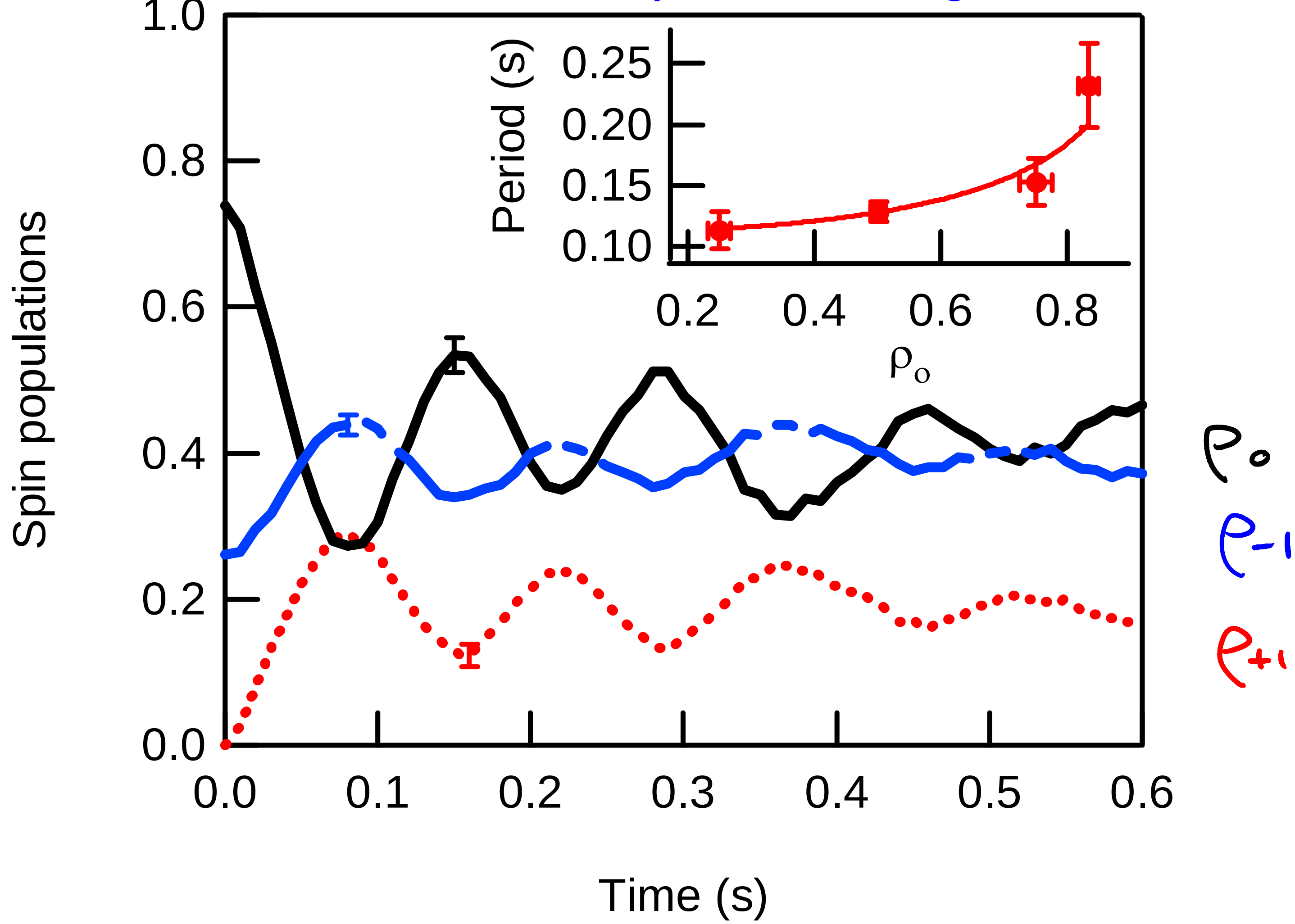


$$2|0\rangle \rightarrow |+\rangle + |-\rangle$$



$$|+\rangle + |-\rangle \rightarrow 2|0\rangle$$

Coherent spin mixing



"Coherent spin dynamics in a spin-1 Bose condensates," M.-S. Chang, Q. Qin, W. Zhang, L. You, and M. S. Chapman, Nature Physics, 1, 111 (2005).

Coherent spin oscillations at high B field

Josephson equations of motion

$$\frac{\partial \rho_0}{\partial t} = \frac{2c}{\hbar}\rho_0\sqrt{1-\rho_0^2-M^2}\sin\theta$$

$$\frac{\partial \theta}{\partial t} = -\frac{2q}{\hbar} + \frac{2c}{\hbar}(1-2\rho_0) + \frac{2c}{\hbar}\frac{(1-\rho_0)(1-2\rho_0)-M^2}{\sqrt{(1-\rho_0)^2-M^2}}\cos\theta$$

Dynamical variables:

$\rho_0(t)$ **: the population of the m = 0 state**

$\theta(t) \equiv \theta_+ + \theta_- - 2\theta_0$ **the relative phase of the spinor components**

Quadratic Zeeman energy $q = (E_{+1} + E_{-1} - 2E_0)/2$

Josephson equations of motion

$$\frac{\partial \rho_0}{\partial t} = \frac{2c}{\hbar}\rho_0\sqrt{1-\rho_0^2-M^2}\sin\theta$$

$$\frac{\partial \theta}{\partial t} = -\frac{2q}{\hbar} + \frac{2c}{\hbar}(1-2\rho_0) + \frac{2c}{\hbar}\frac{(1-\rho_0)(1-2\rho_0)-M^2}{\sqrt{(1-\rho_0)^2-M^2}}\cos\theta$$

for $q \gg c$ the equations reduce to AC Josephson equations $\Longrightarrow$

$$\rho_0(t) = (A/q)\sin 2\delta t$$

$$\theta(t) = -(2q/\hbar)t$$

Dynamical variables:

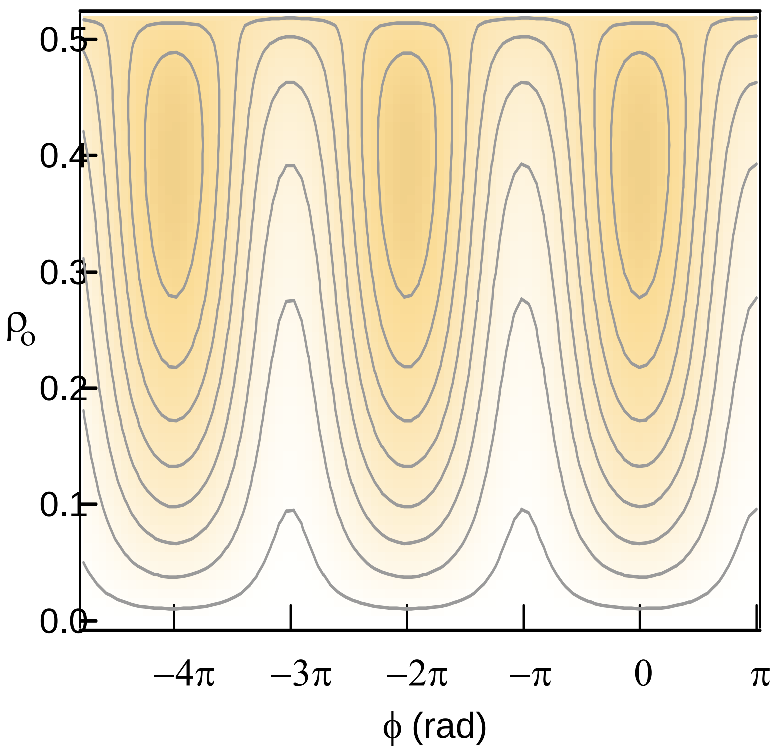
$\rho_0(t)$ **: the population of the $m_F = 0$ state**

$\theta(t) \equiv \theta_+ + \theta_- - 2\theta_0$ **the relative phase of the spinor components**

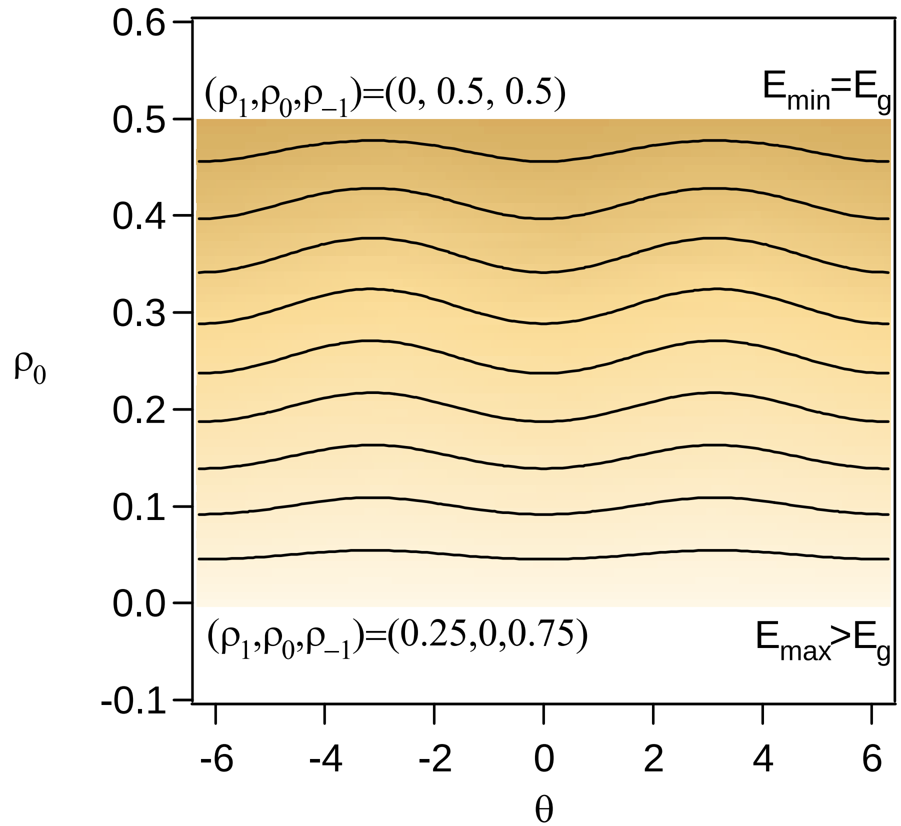
Quadratic Zeeman energy $q = (E_{+1} + E_{-1} - 2E_0)/2$

Energy contours

low field



high field

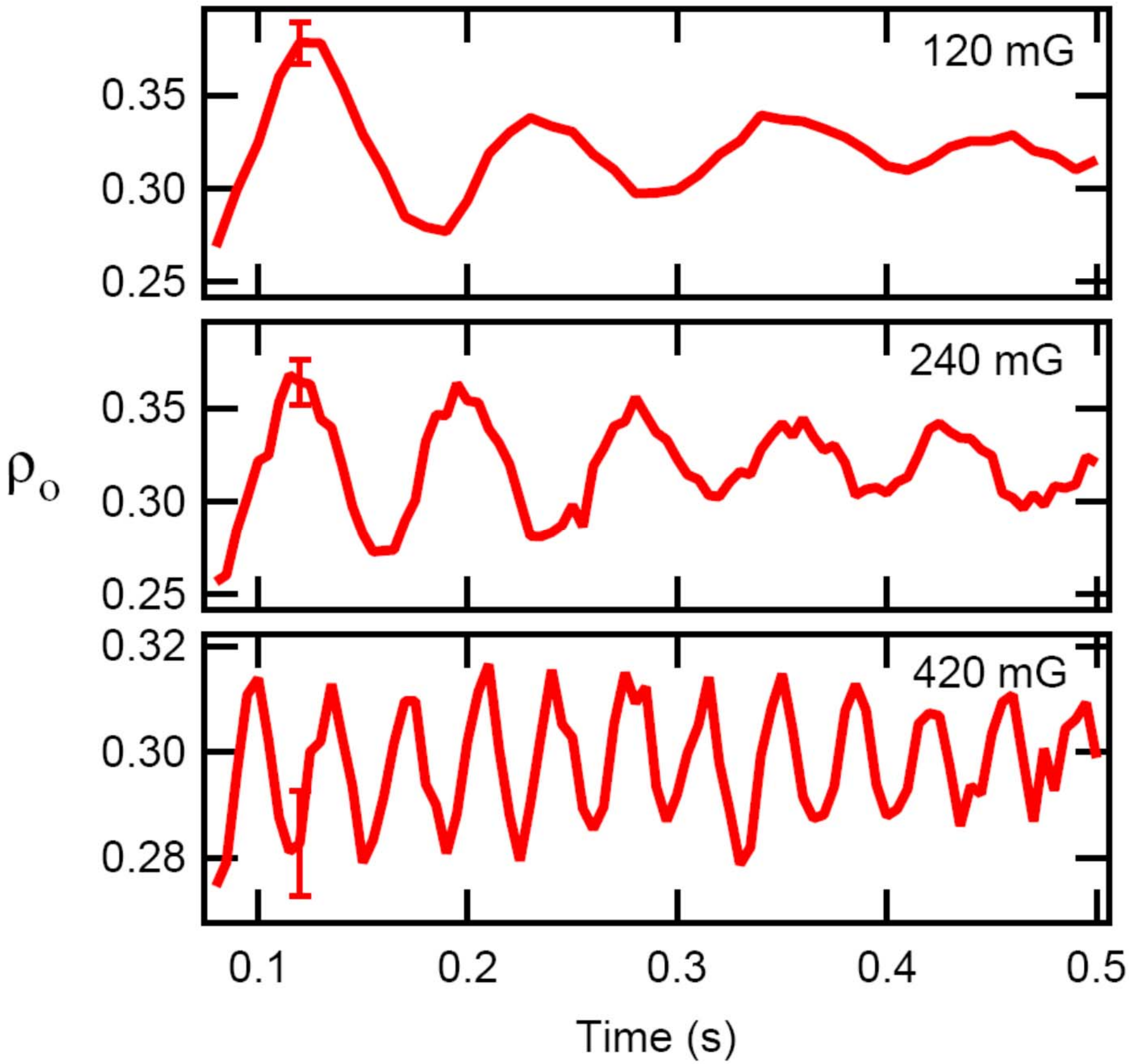


AC Josephson Oscillations

For high fields where q >> c, the system exhibits small oscillations analogous to AC-Josephson oscillations:

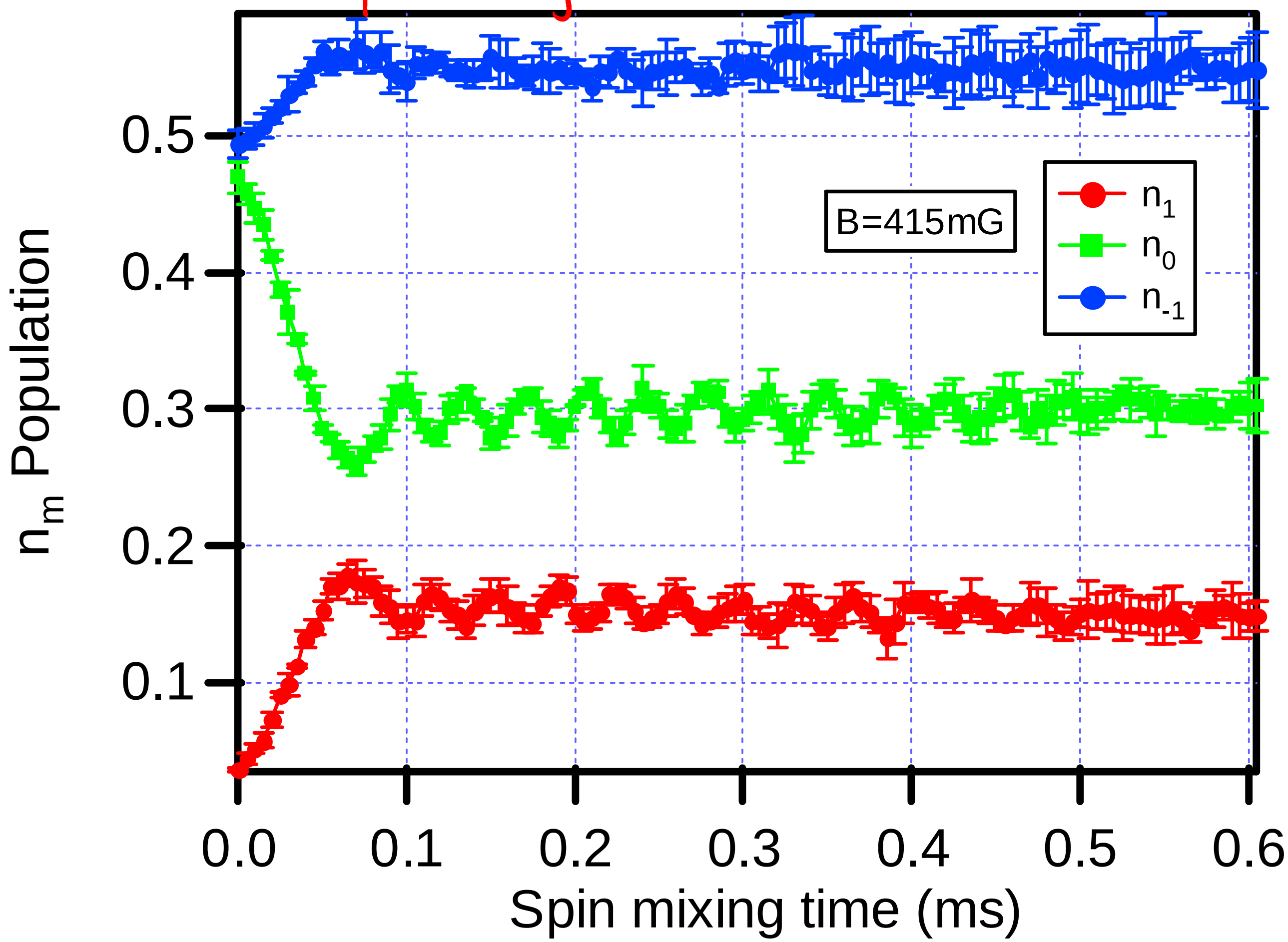
$$\rho_0(t) = (A/q)\sin 2\delta t$$

$$\theta(t) = -(2q/\hbar)t$$



Large field dynamics

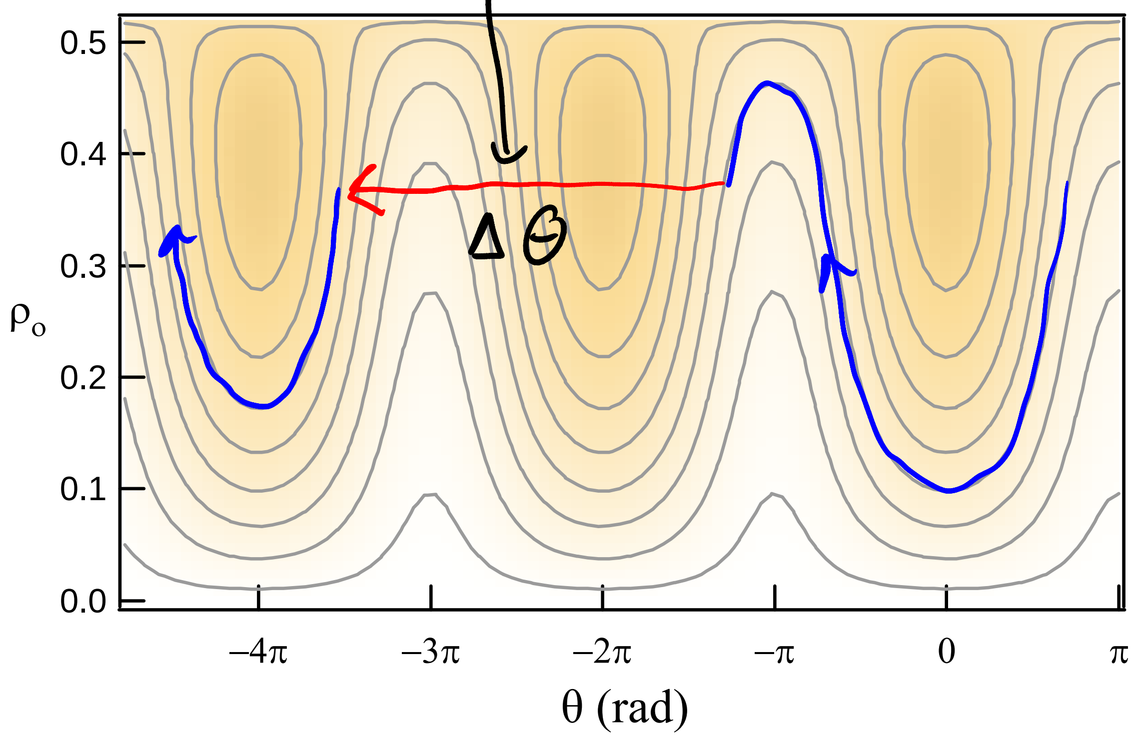
low field ⟵ | ⟶ large field



Coherent control of spin dynamics

Controlling spinor dynamics

pulse on a magnetic field



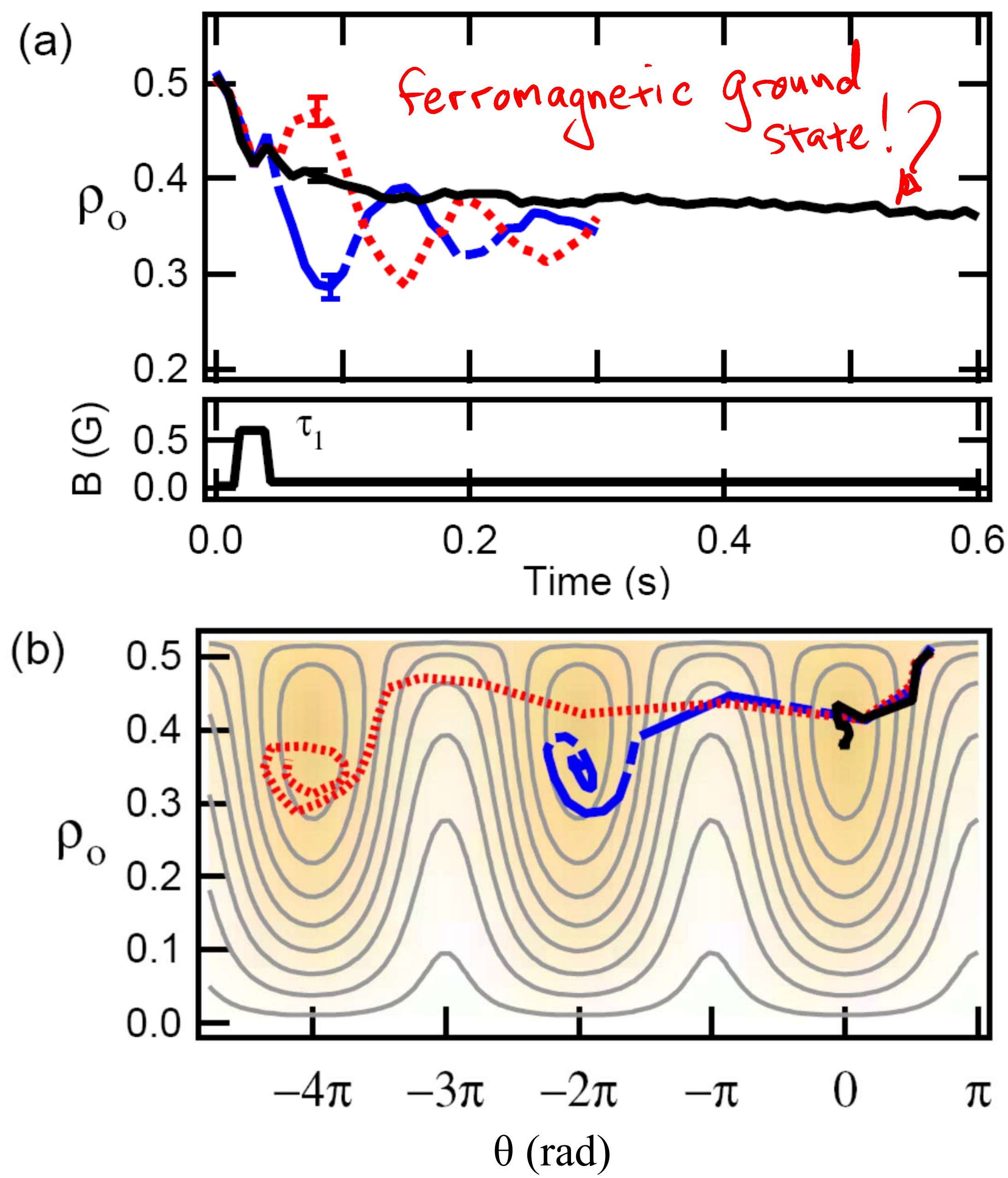
$$\frac{\partial \theta}{\partial t} \approx -\frac{2q}{\hbar}$$

$$\Delta\theta = (2/\hbar)\int \delta dt$$

Can manipulate dynamics by changing the phase in real time via quadratic Zeeman effect

Quadratic Zeeman energy $q = (E_{+1} + E_{-1} - 2E_0)/2$

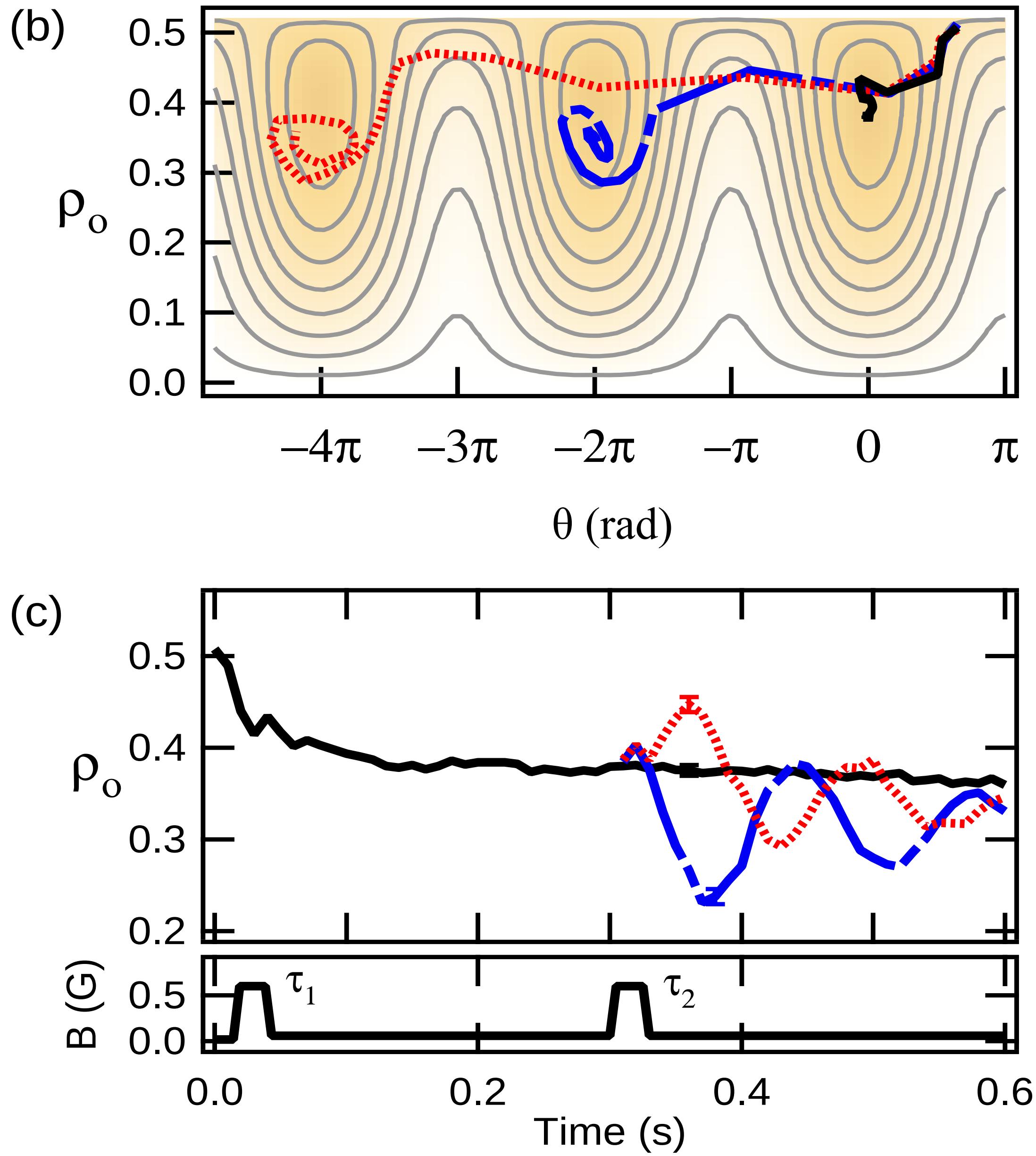
Controlling spinor dynamics



Change trajectories by applying phase shifts via the quadratic zeeman effect

$$\frac{\partial\theta}{\partial t} \approx -\frac{2q}{\hbar}$$

$$\Delta\theta = (2/\hbar)\int q\,dt$$

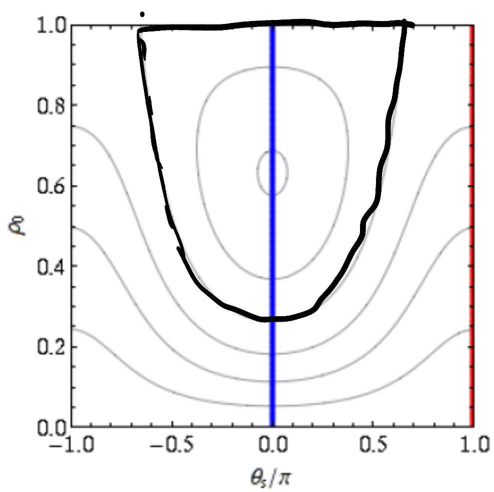


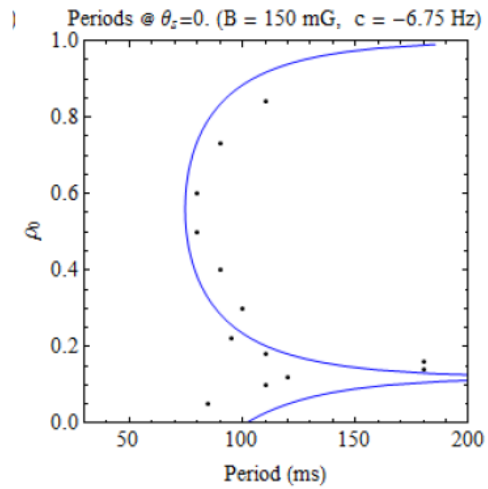
Demonstrating coherence of ferromagnetic ground state

Restarting the dynamics by phase-shifting out of the ground state at a later time

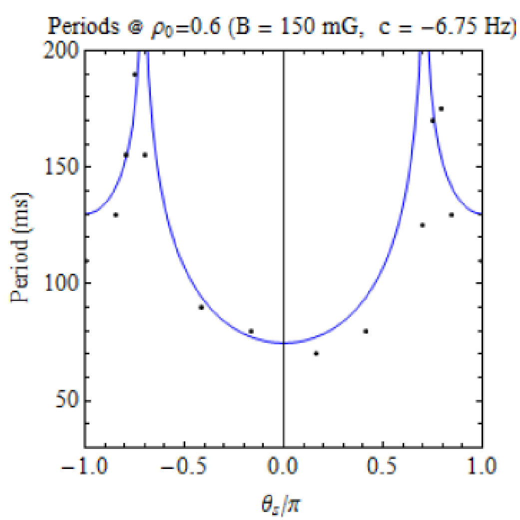
Spin coherence time = condensate lifetime

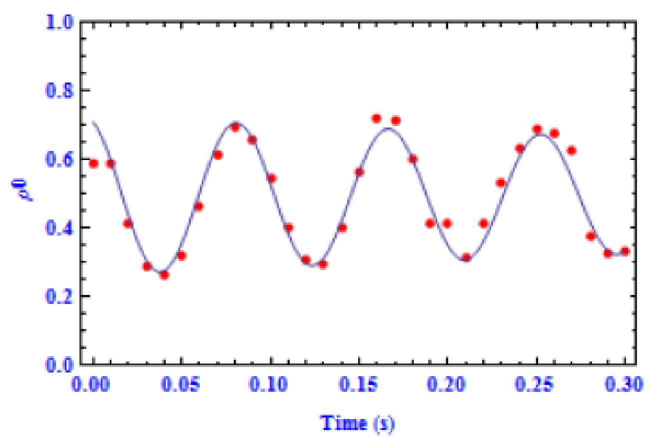
Coherent spin mixing





- Can initialize anywhere with combination of rf pulse and spinor phase shift





Quantum spin dynamics beyond mean-field

Quantum solution for spin-1 system

PHYSICAL REVIEW LETTERS

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Quantum Spins Mixing in Spinor Bose-Einstein Condensates

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(Received 18 May 1998)

A set of collective spin states is derived for a trapped Bose-Einstein condensate in which atoms have three internal hyperfine spins. These collective states minimize the interaction energy among condensate atoms, and they are characterized by strong spin correlations. We also examine the internal dynamics of an initially spin-polarized condensate. The time scale of spin mixing is predicted.
[S0031-9007(98)07921-6]

Quantum solution

Spin part of Hamiltonian

$$H_a = \frac{c_2}{2}\int\Big(\hat{\Psi}_1^\dagger\hat{\Psi}_1^\dagger\hat{\Psi}_1\hat{\Psi}_1 + \hat{\Psi}_{-1}^\dagger\hat{\Psi}_{-1}^\dagger\hat{\Psi}_{-1}\hat{\Psi}_{-1} + 2\hat{\Psi}_1^\dagger\hat{\Psi}_0^\dagger\hat{\Psi}_1\hat{\Psi}_0 + 2\hat{\Psi}_{-1}^\dagger\hat{\Psi}_0^\dagger\hat{\Psi}_{-1}\hat{\Psi}_0 - 2\hat{\Psi}_1^\dagger\hat{\Psi}_{-1}^\dagger\hat{\Psi}_1\hat{\Psi}_{-1}$$
$$+2\hat{\Psi}_1^\dagger\hat{\Psi}_{-1}^\dagger\hat{\Psi}_0\hat{\Psi}_0 + 2\hat{\Psi}_0^\dagger\hat{\Psi}_0^\dagger\hat{\Psi}_1\hat{\Psi}_{-1}\Big)d^3x$$

Single spatial mode approximation $\longrightarrow$ $\hat{\Psi}_\kappa \approx \hat{a}_\kappa\,\phi(\mathbf{r})$

$$H_a = \lambda_a'(a_1^\dagger a_1^\dagger a_1 a_1 + a_{-1}^\dagger a_{-1}^\dagger a_{-1} a_{-1} - 2a_1^\dagger a_{-1}^\dagger a_1 a_{-1} + 2a_1^\dagger a_0^\dagger a_1 a_0 + 2a_{-1}^\dagger a_0^\dagger a_{-1} a_0$$
$$+2a_0^\dagger a_0^\dagger a_1 a_{-1} + 2a_1^\dagger a_{-1}^\dagger a_0 a_0)$$

$$2\lambda_i' = \lambda_i \int |\phi(\mathbf{r})^4| d^3r$$

Introduce angular momentum operators

$$S_- \equiv \sqrt{2}\left(a_1^\dagger a_0 + a_0^\dagger a_{-1}\right) \quad S_+ \equiv \sqrt{2}\left(a_0^\dagger a_1 + a_{-1}^\dagger a_0\right) \quad S_z \equiv \sqrt{2}\left(a_1^\dagger a_1 - a_{-1}^\dagger a_{-1}\right)$$

$$\boxed{H_a = \lambda_a' S^2}$$

Signatures of quantum spin-dynamics

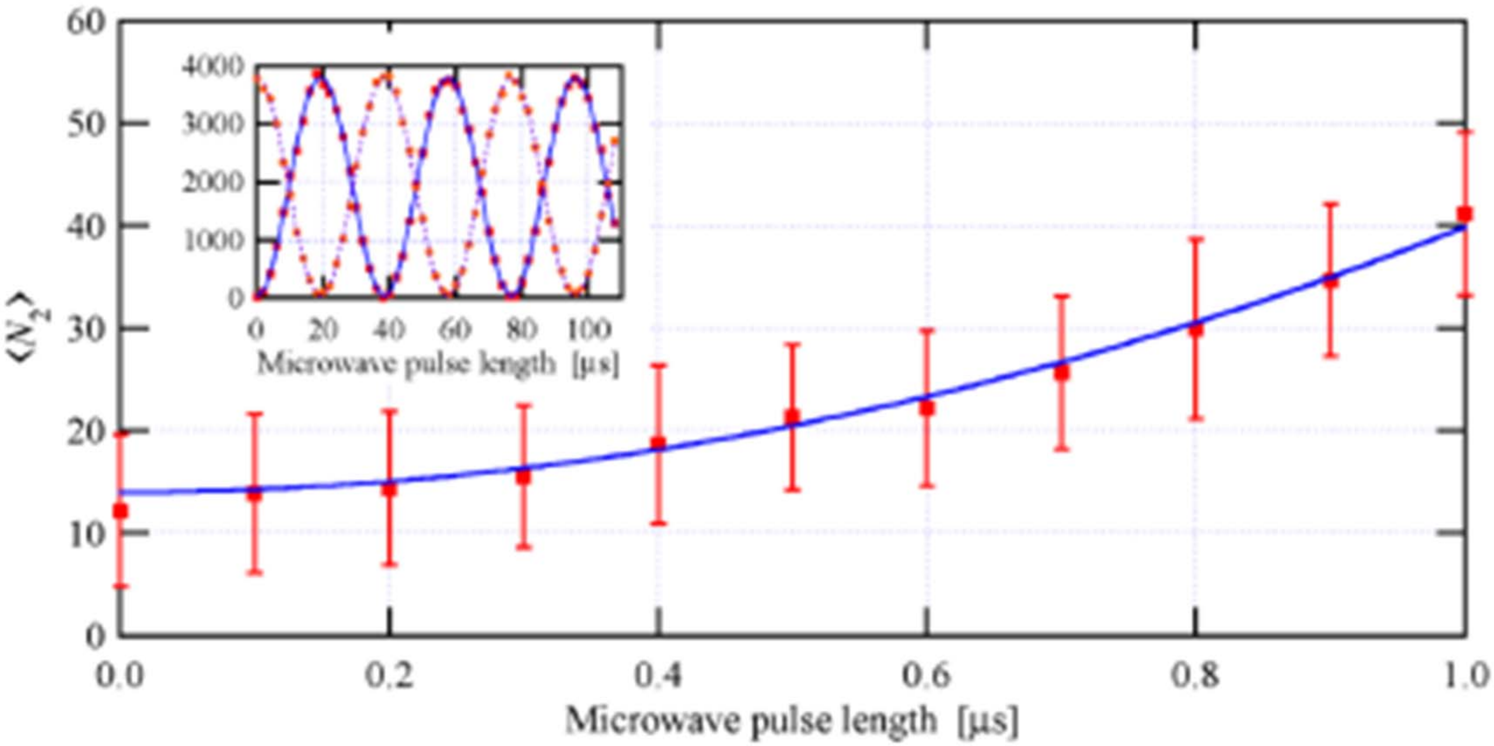
Characteristics

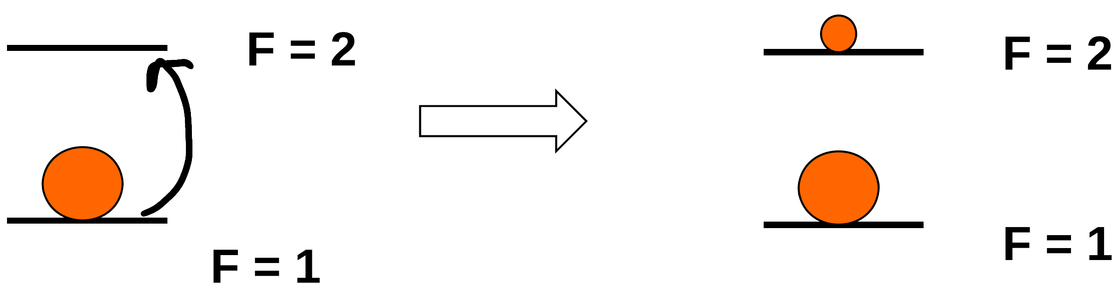
- Non-linear evolution of spin populations
- Non-poissonian fluctuations

  Squeezing, and super-poissonian noise

Measuring small atom numbers

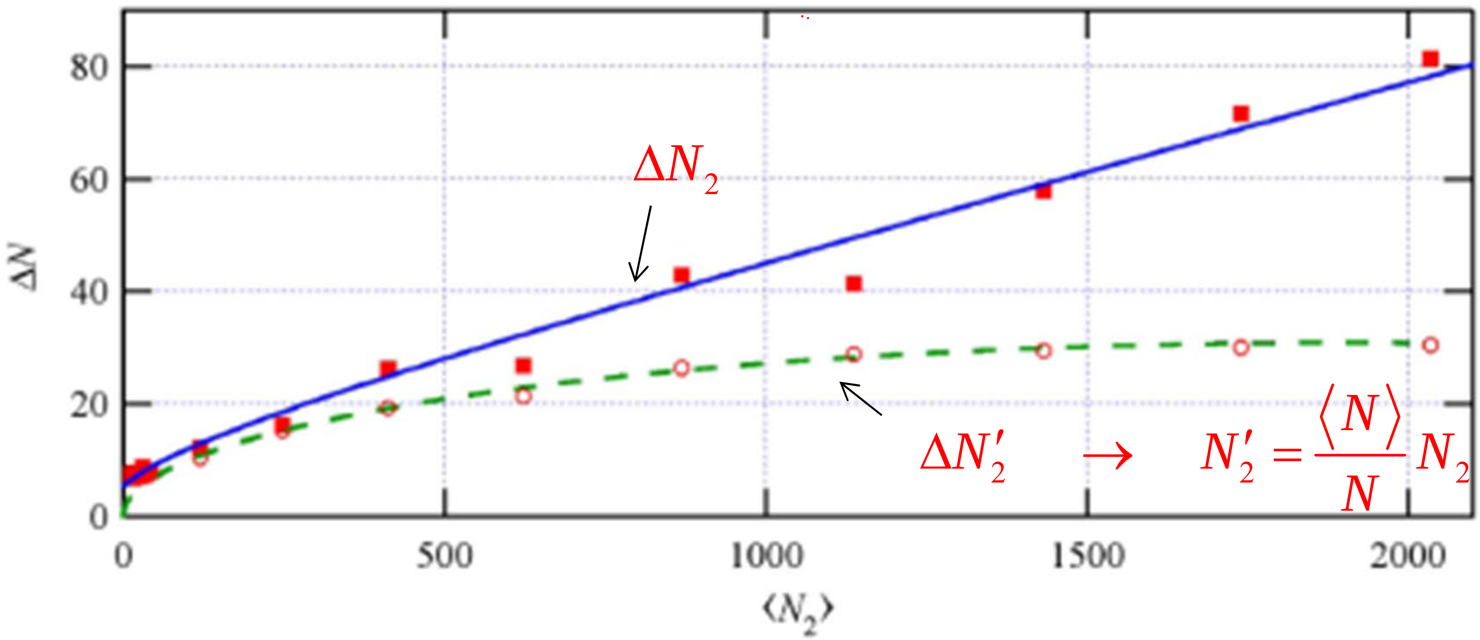
Fluorescent imaging of small numbers of atoms in F = 2 state

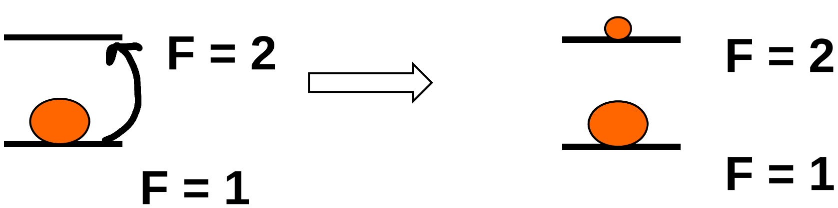




Demonstrating shot noise limited detection

Same as before, but now examining the noise

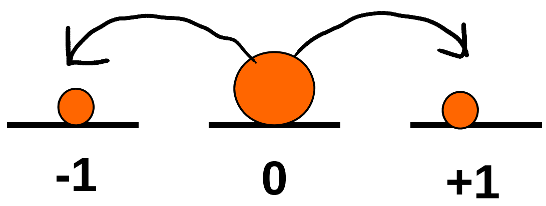


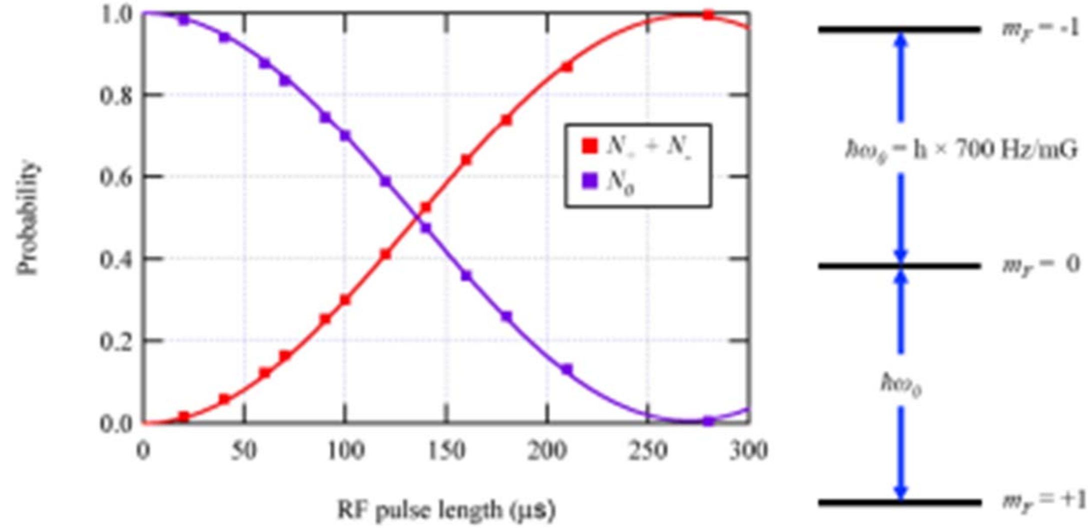


Correcting for the variation in the total number of atoms reveals shot-noise limited detection

Shot-noise limited measurement of magnetization

Rf Rabi oscillations in F = 1 manifold



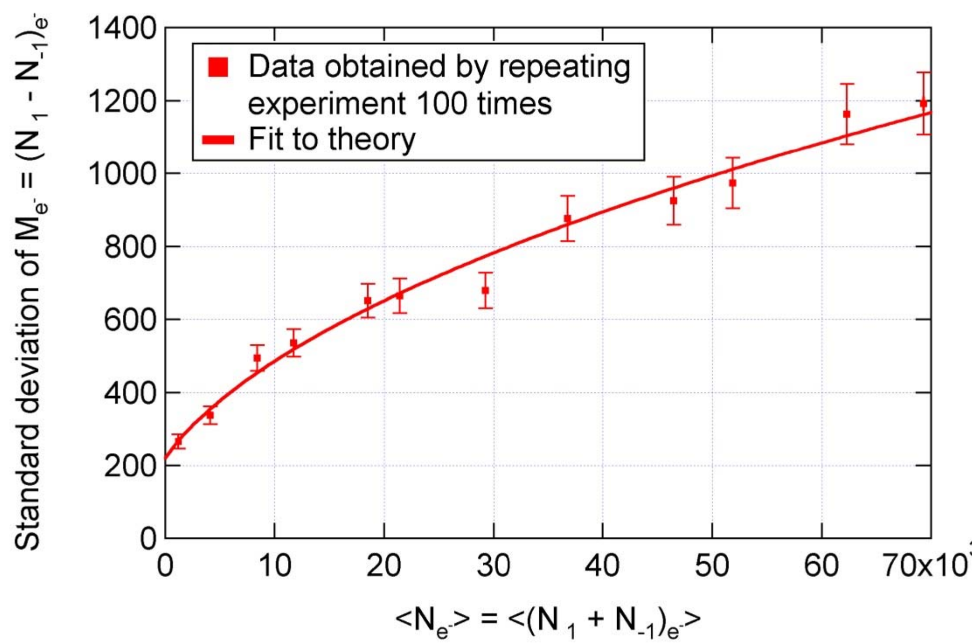


Magnetization

$$M = N_+ - N_-$$

$$\Delta M = \Delta\left(N_{+1} - N_{-1}\right)$$

$$\left(\Delta M\right)^2 = N_+ + N_- = N\sin^2(\Omega t)$$



Number squeezing

$$\hat{H}_{4WSM} = \lambda(\hat{a}_0^2 \hat{a}_{+1}^\dagger \hat{a}_{-1}^\dagger + \hat{a}_0^{\dagger 2} \hat{a}_{+1} \hat{a}_{-1})$$

The spin mixing Hamiltonian is an analogue to optical 4 wave mixing

- 4 wave spin mixing

Atoms in the +1 and -1 states are generated with equal numbers

Starting with a pure $m_F = 0$ condensate, the magnetization is conserved

$$M = N_{+1} - N_{-1} = 0$$

Importantly, the uncertainty is also zero

$$\Delta M = 0$$

- This a form of relative number squeezing

Number squeezing

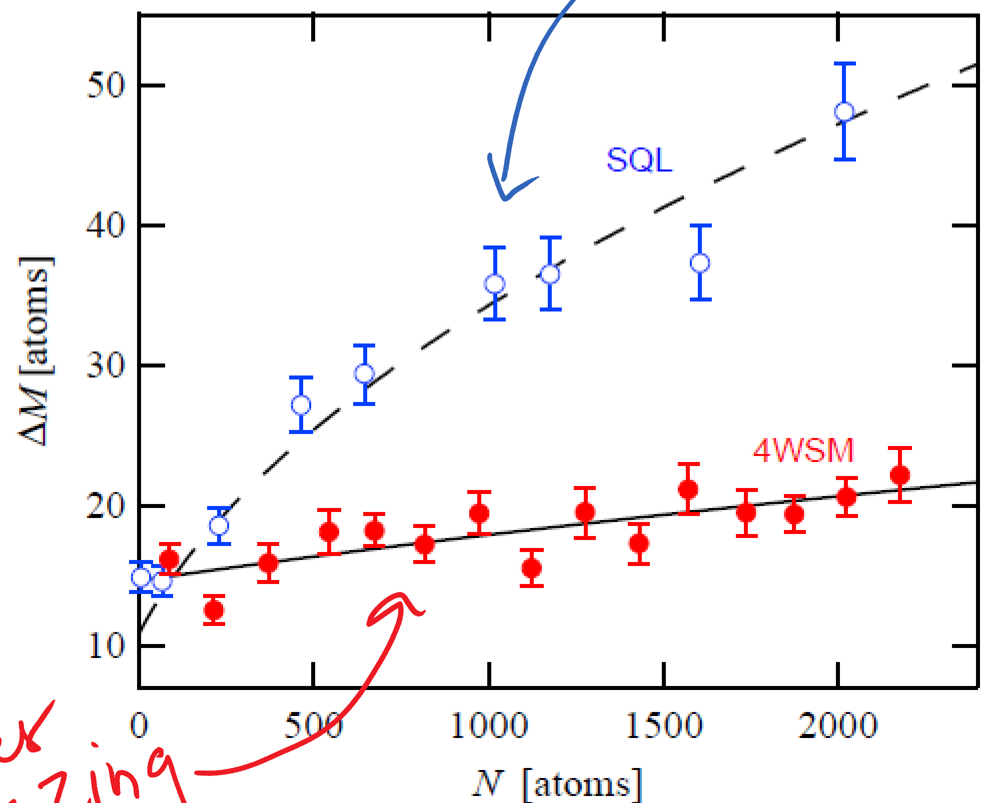
$$\hat{H}_{4WSM} = \lambda\left(\hat{a}_0^2 \hat{a}_{+1}^\dagger \hat{a}_{-1}^\dagger + \hat{a}_0^{\dagger 2} \hat{a}_{+1} \hat{a}_{-1}\right)$$

- 4 wave spin mixing

- Spin mixing from a pure mF = 0 state
- Measured M
- Repeat to collect statistics

$$M = N_{+1} - N_{-1} = 0$$

coherent state



number squeezing

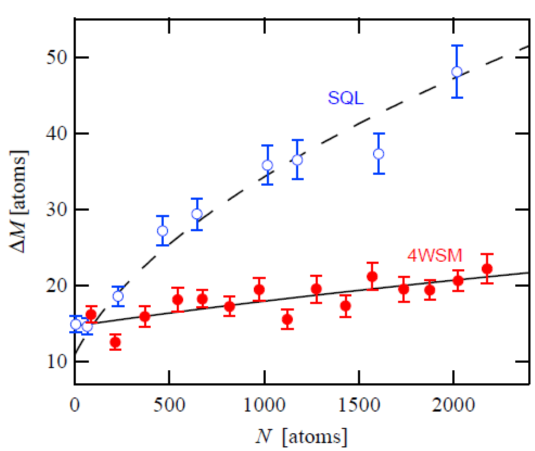
E. Bookjans et al, PRL 2011

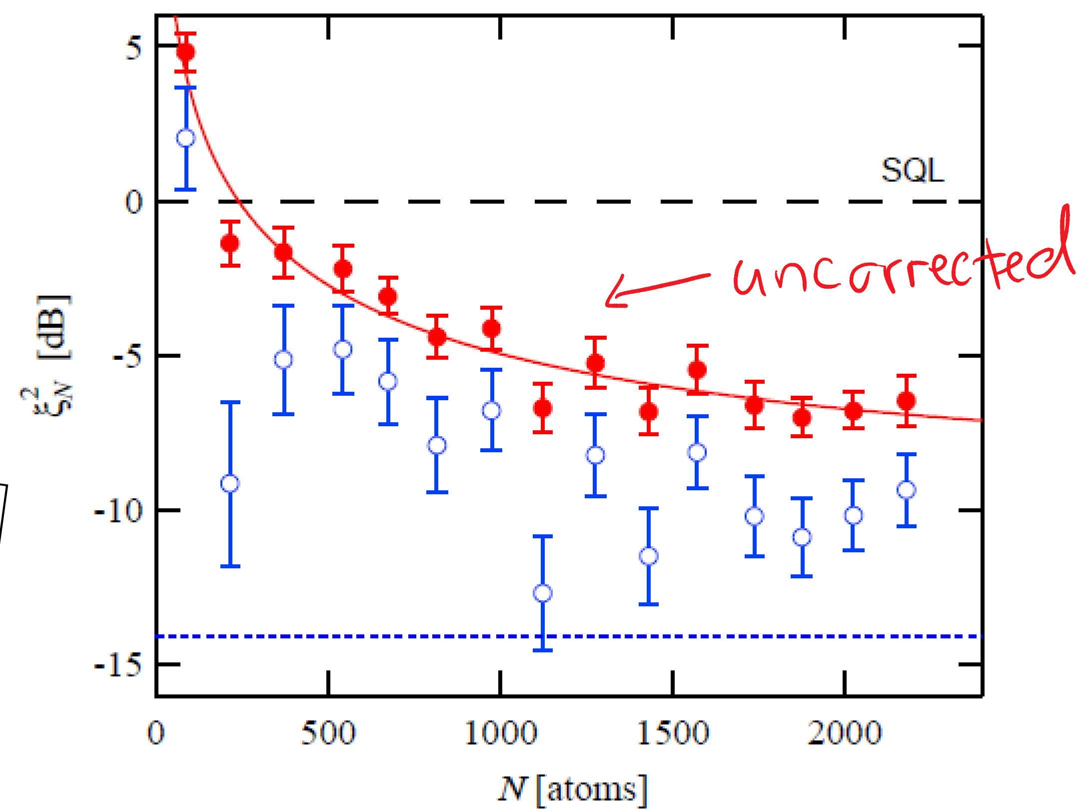
Number squeezing

$$\hat{H}_{4WSM} = \lambda(\hat{a}_0^2 \hat{a}_{+1}^\dagger \hat{a}_{-1}^\dagger + \hat{a}_0^{\dagger 2} \hat{a}_{+1} \hat{a}_{-1})$$

- Convert into a squeezing parameter

$$\xi_N^2 = \Delta M^2 / N$$





Spin-nematic quadrature squeezing

Squeezing in a spin-1 system

Spin-1/2

- can be fully described using just the collective spin vector (3-components)

  SU(2)
- Can visualize on single Bloch sphere

Higher spin systems require additional degrees of freedom

- For spin-1, the quadrupole (nematic) tensor is natural d.o.f.
- Traceless, symmetric, 5 independent components

  SU(3)

$$Q_{ij} = S_i S_j + S_j S_i - \frac{4}{3}\delta_{ij}$$

$$\hat{Q}_{ij} \; (\{i,j\} \in \{x,y,z\}$$

- Need multiple Bloch spheres to visualize

Spin vector

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\hat{S}_x = \frac{1}{\sqrt{2}} \left(\hat{a}_1^\dagger \hat{a}_0 + \hat{a}_0^\dagger \hat{a}_{-1} + \hat{a}_0^\dagger \hat{a}_1 + \hat{a}_{-1}^\dagger \hat{a}_0 \right)$$

$$S_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\hat{S}_y = \frac{i}{\sqrt{2}} \left(-\hat{a}_1^\dagger \hat{a}_0 - \hat{a}_0^\dagger \hat{a}_{-1} + \hat{a}_0^\dagger \hat{a}_1 + \hat{a}_{-1}^\dagger \hat{a}_0 \right)$$

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\hat{S}_z = \left(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_{-1}^\dagger \hat{a}_{-1} \right)$$

Quadrupole (nematic) tensor

$$Q_{yz} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \hat{Q}_{yz} = \frac{i}{\sqrt{2}} \left(-\hat{a}_1^\dagger \hat{a}_0 + \hat{a}_0^\dagger \hat{a}_{-1} + \hat{a}_0^\dagger \hat{a}_1 - \hat{a}_{-1}^\dagger \hat{a}_0 \right)$$

$$Q_{xz} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \hat{Q}_{xz} = \frac{1}{\sqrt{2}} \left(\hat{a}_1^\dagger \hat{a}_0 - \hat{a}_0^\dagger \hat{a}_{-1} + \hat{a}_0^\dagger \hat{a}_1 - \hat{a}_{-1}^\dagger \hat{a}_0 \right)$$

$$Q_{xy} = i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \hat{Q}_{xy} = i \left(-\hat{a}_1^\dagger \hat{a}_{-1} + \hat{a}_{-1}^\dagger \hat{a}_1 \right)$$

$$Q_{xx} = \begin{pmatrix} -\frac{1}{3} & 0 & 1 \\ 0 & \frac{2}{3} & 0 \\ 1 & 0 & -\frac{1}{3} \end{pmatrix} \qquad \hat{Q}_{xx} = -\frac{1}{3} \hat{a}_{+1}^\dagger \hat{a}_{+1} + \frac{2}{3} \hat{a}_0^\dagger \hat{a}_0 - \frac{1}{3} \hat{a}_{-1}^\dagger a_{-1} + \hat{a}_{+1}^\dagger \hat{a}_{-1} + \hat{a}_{-1}^\dagger \hat{a}_{+1}$$

$$Q_{yy} = \begin{pmatrix} -\frac{1}{3} & 0 & -1 \\ 0 & \frac{2}{3} & 0 \\ -1 & 0 & -\frac{1}{3} \end{pmatrix} \qquad \hat{Q}_{yy} = -\frac{1}{3} \hat{a}_{+1}^\dagger \hat{a}_{+1} + \frac{2}{3} \hat{a}_0^\dagger \hat{a}_0 - \frac{1}{3} \hat{a}_{-1}^\dagger \hat{a}_{-1} - \hat{a}_{+1}^\dagger \hat{a}_{-1} - \hat{a}_{-1}^\dagger \hat{a}_{+1}$$

$$Q_{zz} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix} \qquad \hat{Q}_{zz} = \frac{2}{3} \hat{a}_{+1}^\dagger \hat{a}_{+1} - \frac{4}{3} \hat{a}_0^\dagger \hat{a}_0 + \frac{2}{3} \hat{a}_{-1}^\dagger \hat{a}_{-1}$$

Quantum Hamiltonian in S-N operators

$$\mathcal{H} = \lambda[(2\hat{N}_0 - 1)(\hat{N}_1 + \hat{N}_{-1}) + 2\hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_{-1} + 2\hat{a}_1^\dagger \hat{a}_{-1}^\dagger \hat{a}_0 \hat{a}_0] + q(\hat{N}_1 + \hat{N}_{-1}).$$

$$\Downarrow$$

$$\mathcal{H} = \lambda \hat{S}^2 + \frac{1}{2} q \hat{Q}_{zz}$$

$$\hat{Q}_{zz} = -2\hat{N}_0 + const$$

Fock basis $\left| N_{+1}, N_0, N_{-1} \right\rangle$

Spin-1 visualization

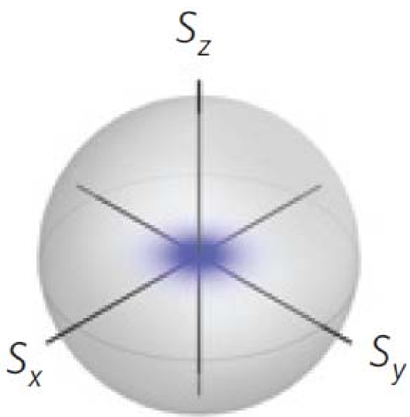
Pure m_F = 0 condensate

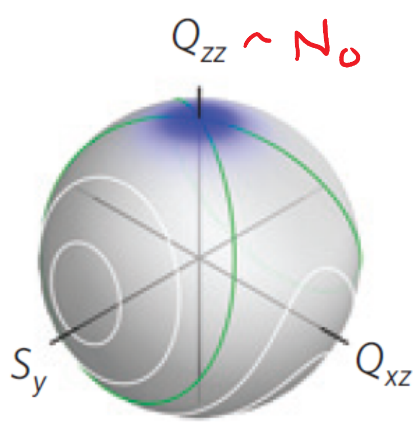
- Fock state

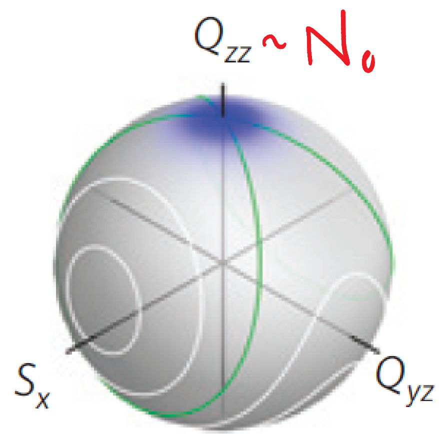
$$|N_{+1}, N_0, N_{-1}\rangle = |0, N, 0\rangle$$

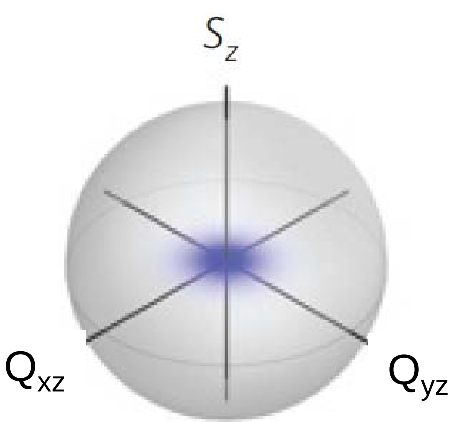
$$\langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = 0$$

$$\langle S_x^2 \rangle = \langle S_y^2 \rangle \neq \langle S_z^2 \rangle$$



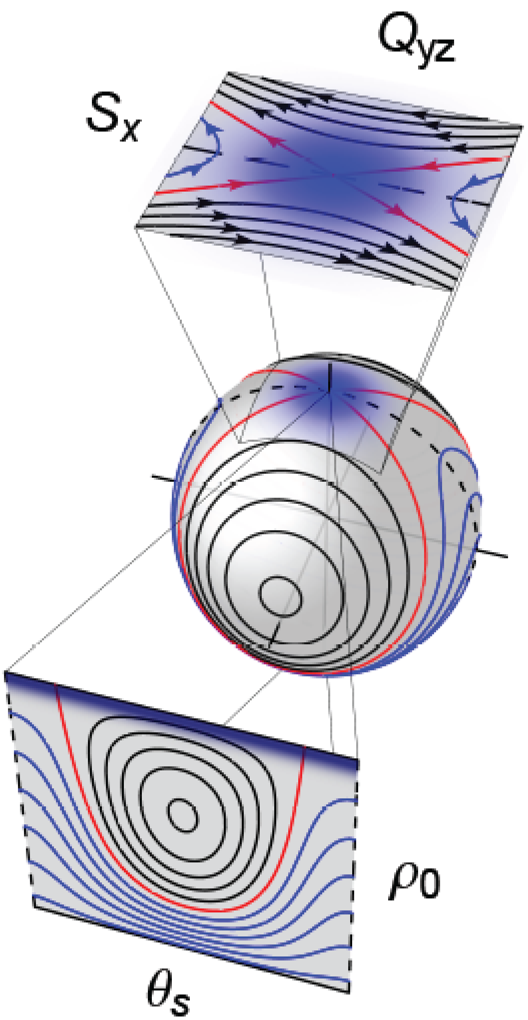






Spin Hamiltonian including magnetic interaction

Relating the phase spaces



$$\mathcal{H} = \lambda \hat{S}^2 + \frac{1}{2} q \hat{Q}_{zz}$$

$$\mathcal{E} \;=\; c\rho_0 \left(1-\rho_0\right)\left(1+\cos\theta_s\right) + q(1-\rho_0)$$

$$\rho_0 = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{8}\left(\left(\frac{S_x + Q_{xz}}{\cos\chi_+}\right)^2 + \left(\frac{S_x - Q_{xz}}{\cos\chi_-}\right)^2\right)}$$

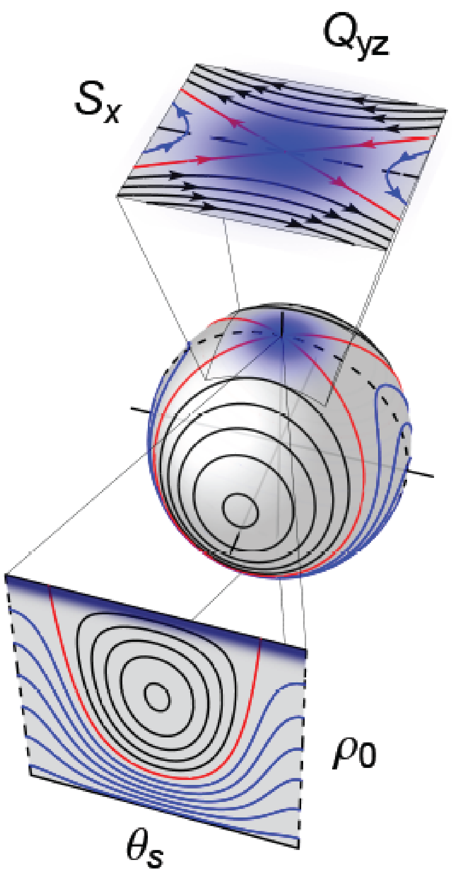
$$\chi_\pm = \arctan \mp \frac{S_y \pm Q_{yz}}{S_x \pm Q_{xz}}$$

$$\chi_\pm = \theta_{\pm 1} - \theta_0$$

Squeezing evolution

Evolution near a hyperbolic fixed point leads to squeezing

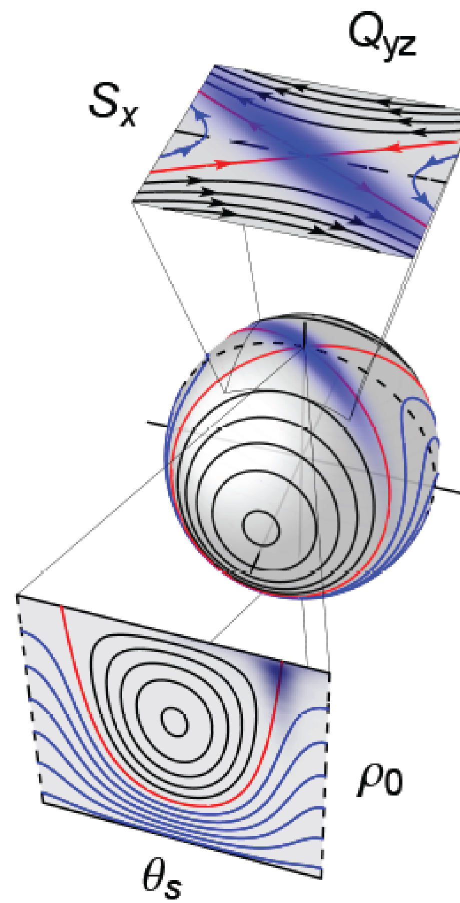
Initial state



Evolution

$$\mathcal{H} = \lambda \hat{S}^2 + \frac{1}{2} q \hat{Q}_{zz}$$

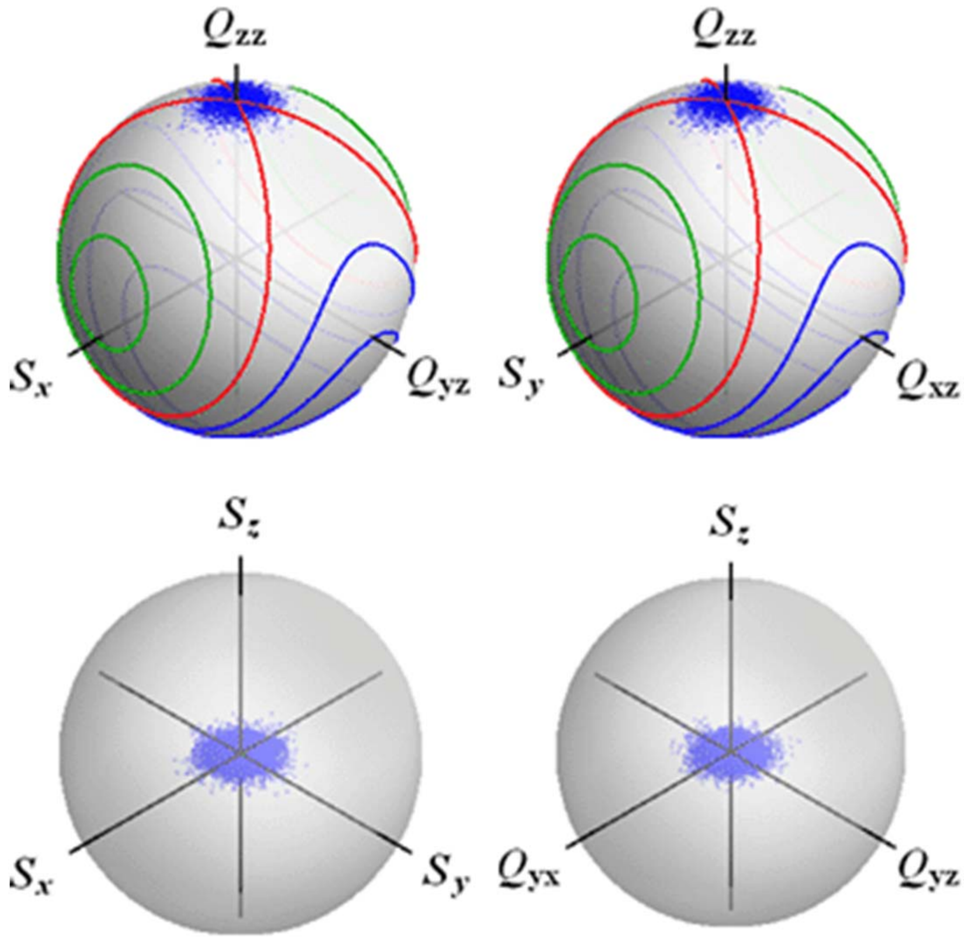
Squeezed state



Squeezing evolution

Evolution near a hyperbolic fixed point leads to squeezing

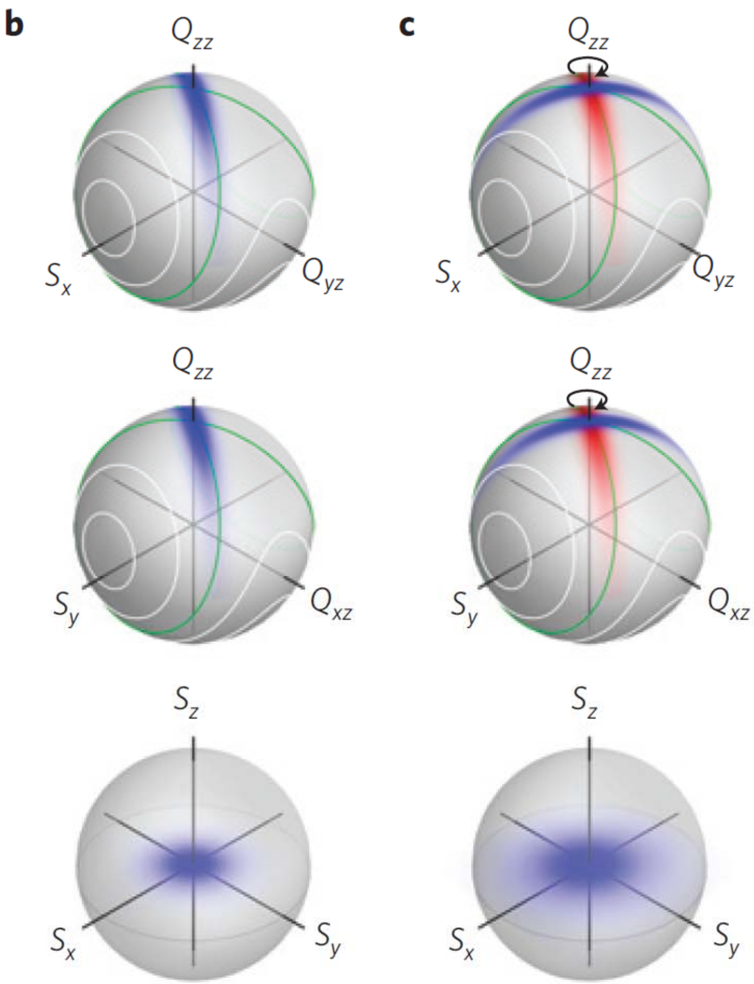
$$\mathcal{H} = \lambda \hat{S}^2 + \frac{1}{2} q \hat{Q}_{zz}$$



The squeezing is not in the spin space so how do we measure it?

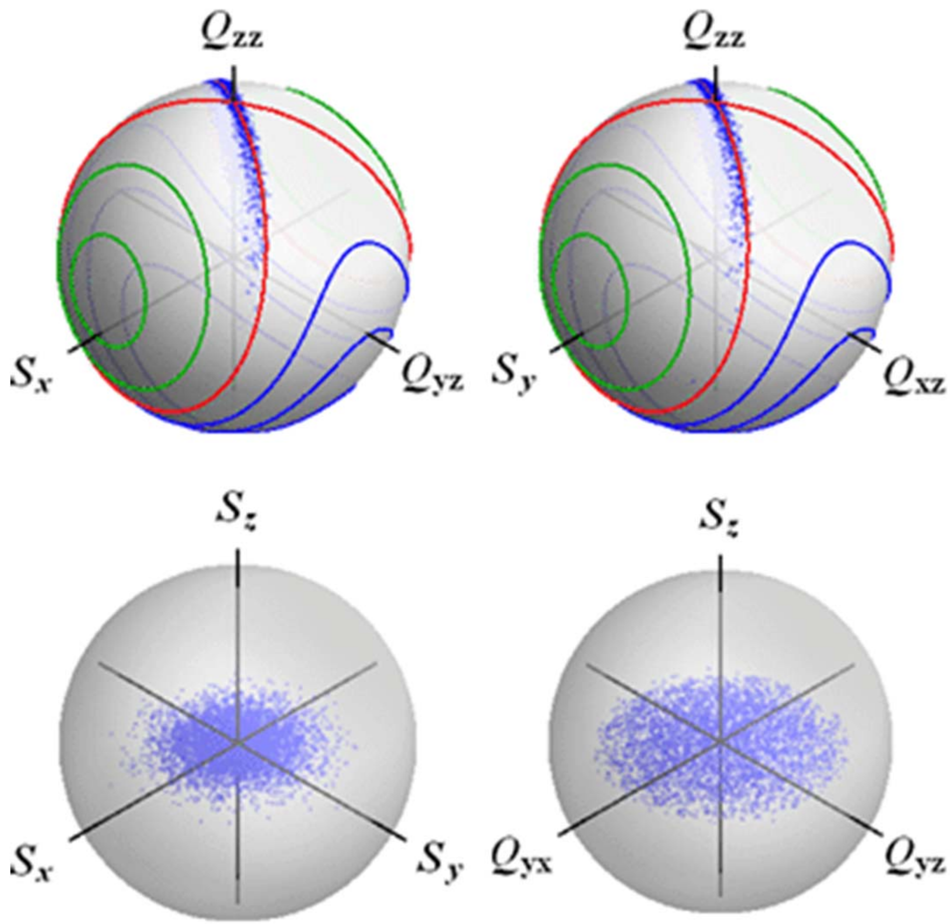
Spin-nematic quadrature rotations

$$\mathcal{H} = \lambda \hat{S}^2 + \frac{1}{2} q \hat{Q}_{zz}$$



- **Can rotate the state about Q_{zz} by pulsing q >> λ briefly**
- **In practice, we use a F = 1 → F =2 microwave Rabi pulse to advance the spinor phase**
- **Different rotations access different quadrature**

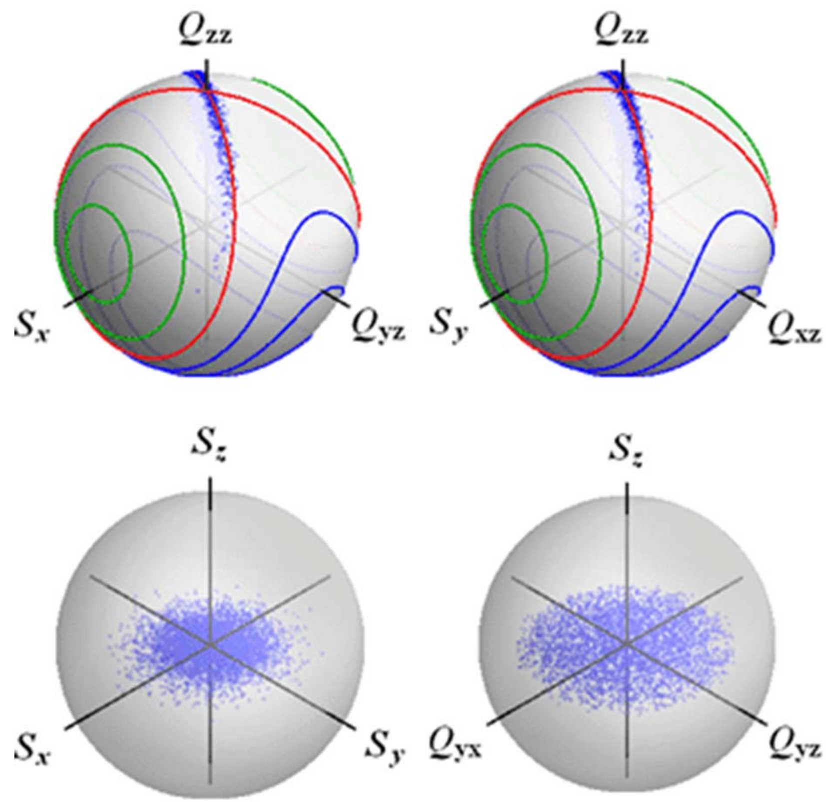
Quadrature rotations



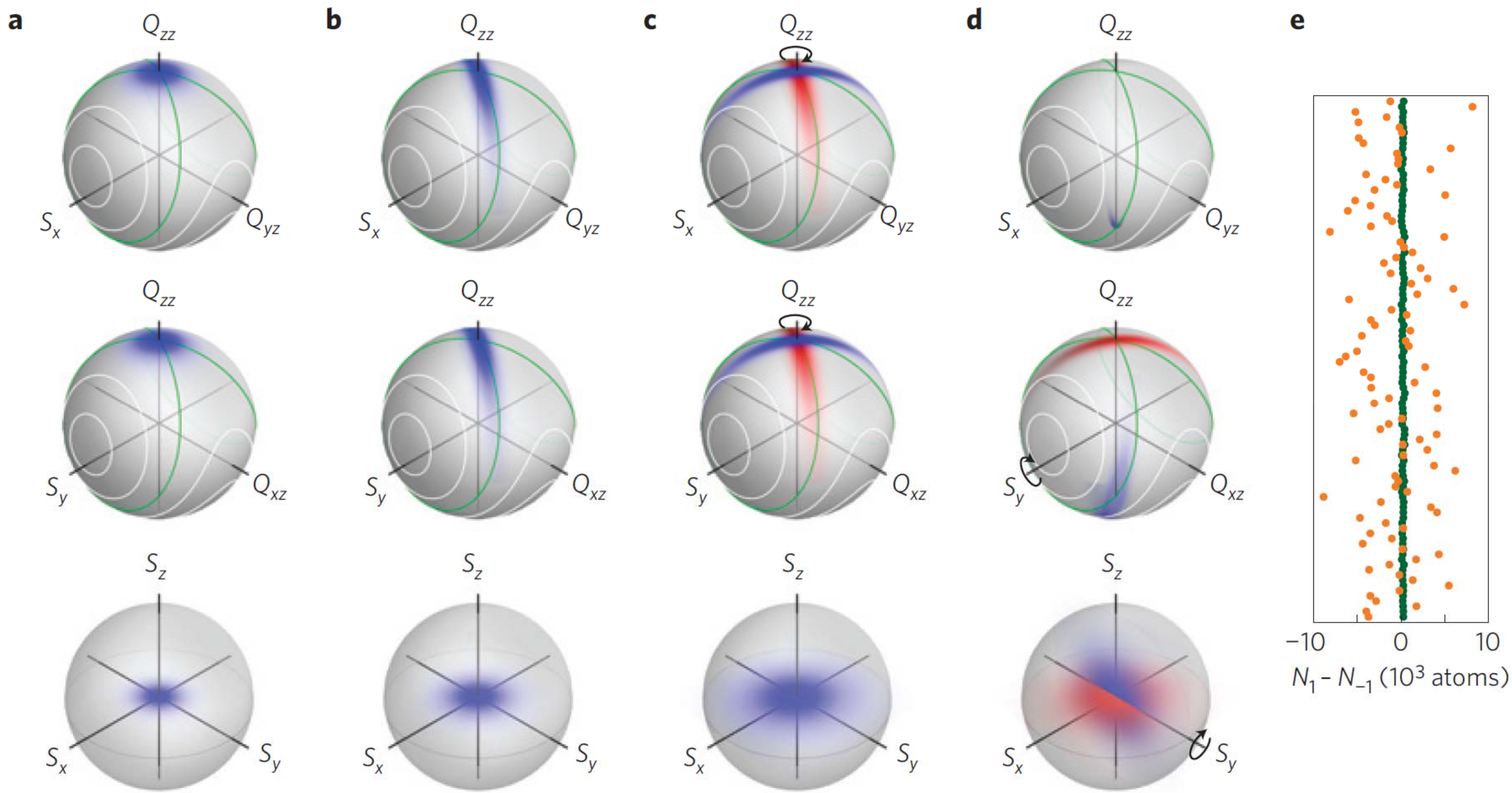
Noise measurement

- Finally, Sx is rotated into Sz for easy measurement
- RF rotation on F = 1 state
- Expansion of condensate in Stern-Gerlach to measure magnetization

$$M = \langle S_z \rangle = N_{+1} - N_{-1}$$

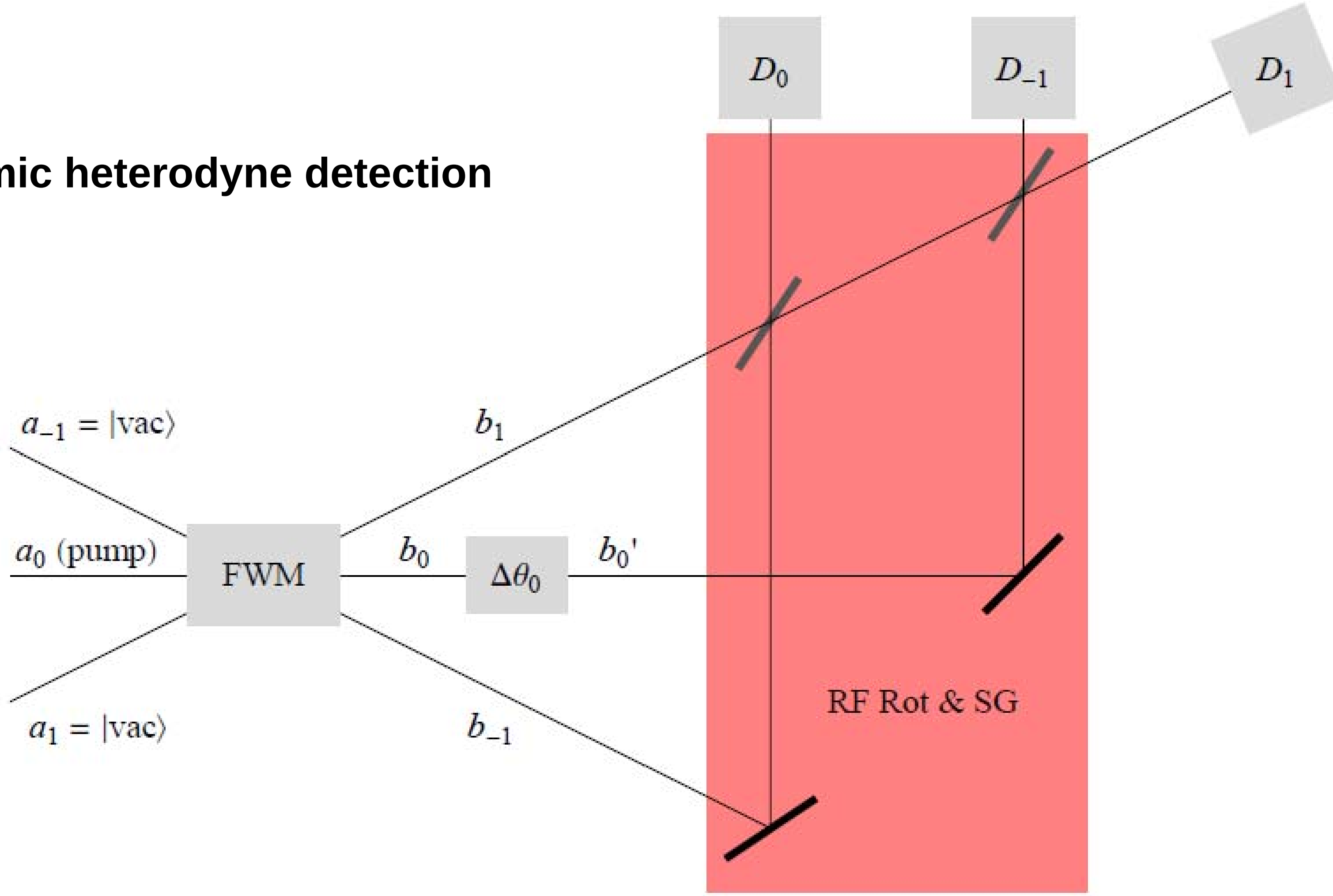


Full sequence

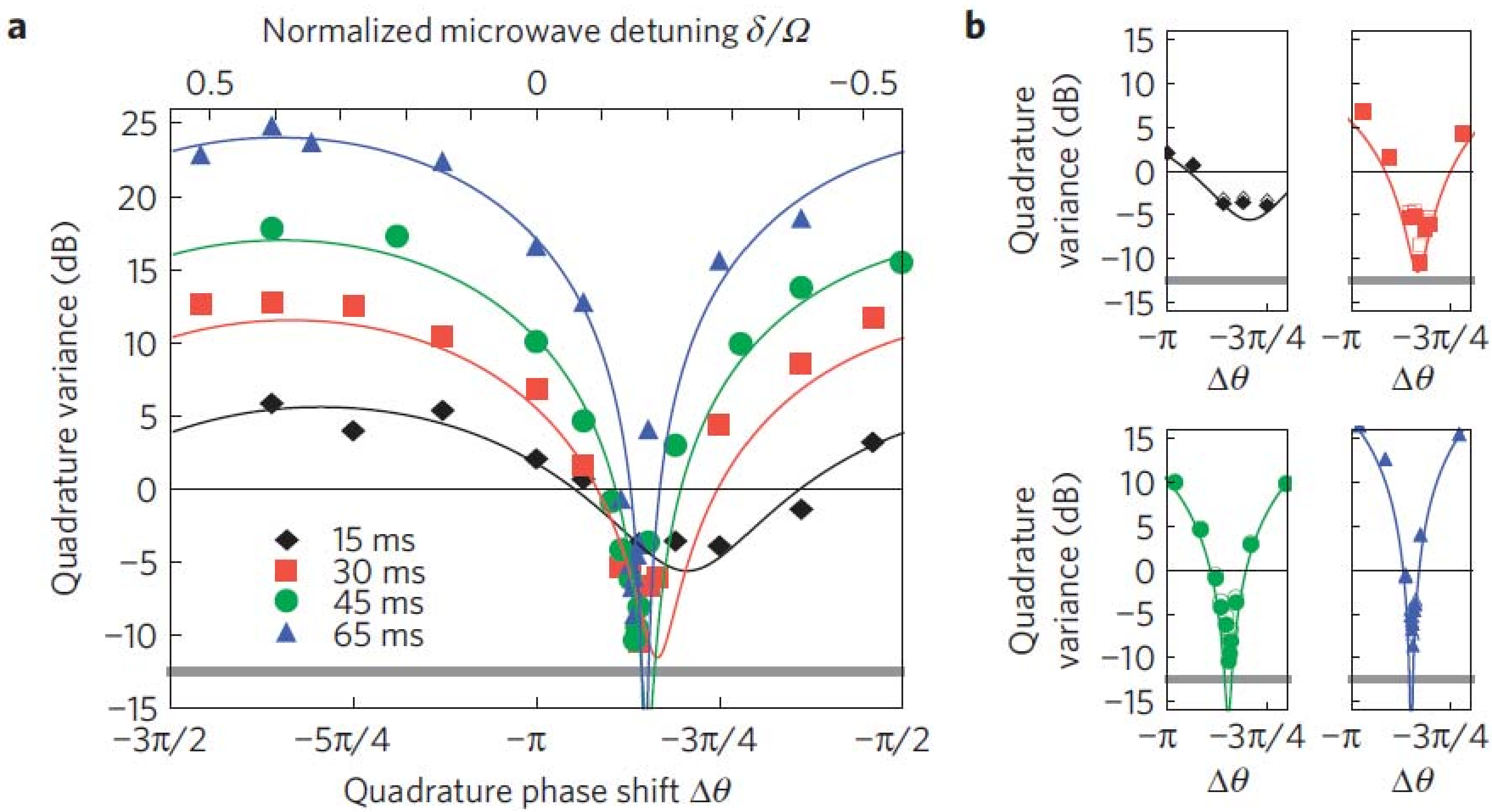


FWM picture

Atomic heterodyne detection



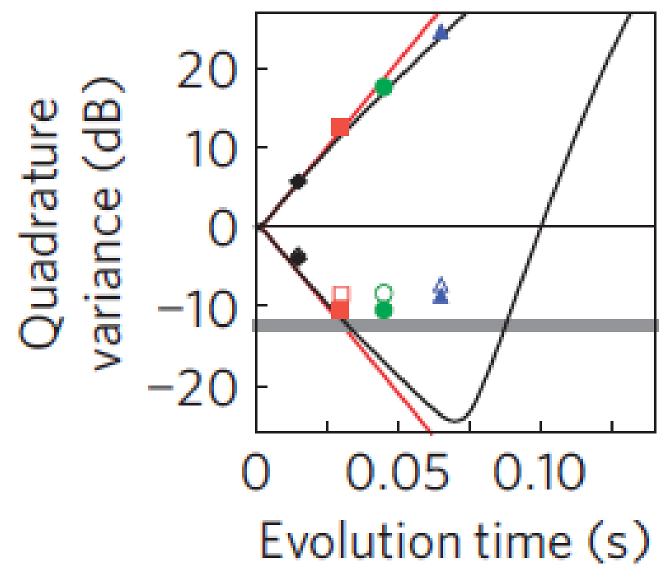
Results

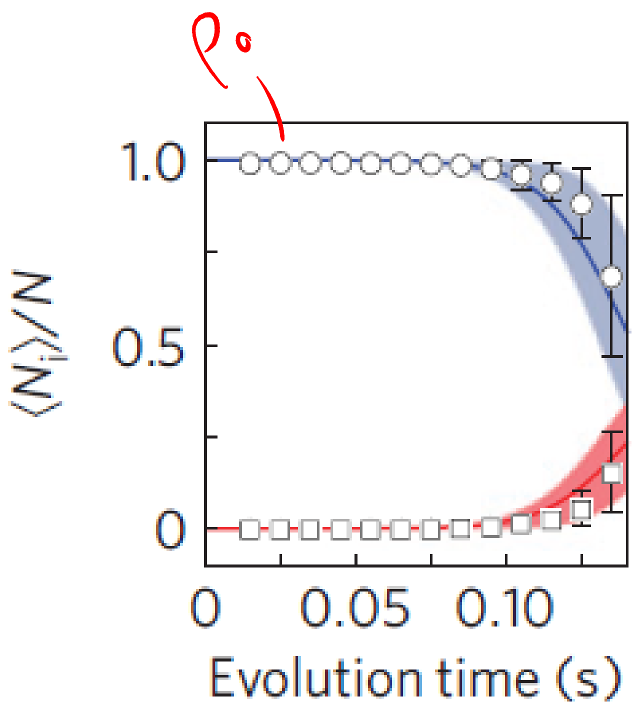


a
Normalized microwave detuning δ/Ω
0.5
0
−0.5
25
20
15
10
5
0
−5
−10
−15
Quadrature variance (dB)
15 ms
30 ms
45 ms
65 ms
−3π/2
−5π/4
−π
−3π/4
−π/2
Quadrature phase shift $\Delta\theta$
b
Quadrature variance (dB)
−π
−3π/4
$\Delta\theta$

Results

Exponential growth of squeezing and anti-squeezing



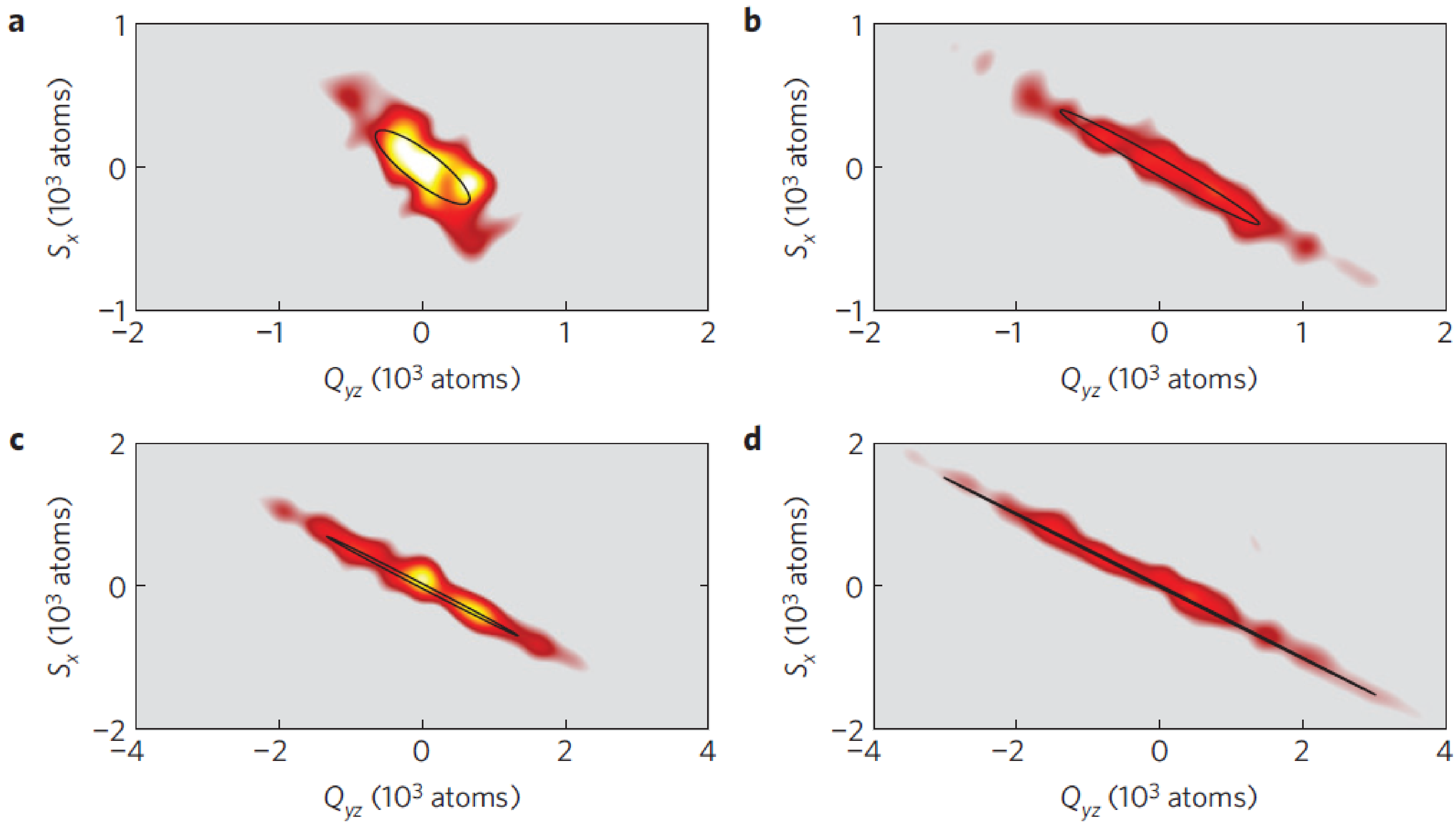


Vacuum squeezing:

no apparent population evolution from initial state at 50 ms

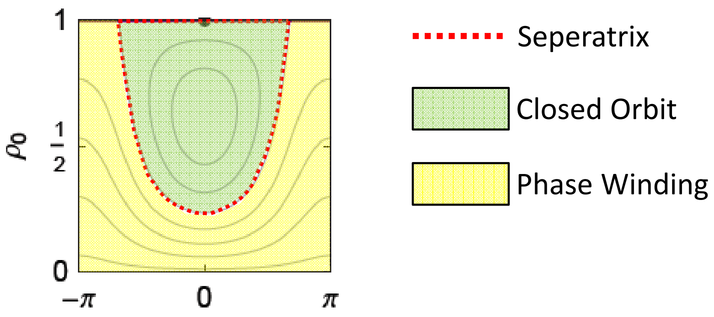
Tomography

Inverse Radon transformation to reconstruct phase space

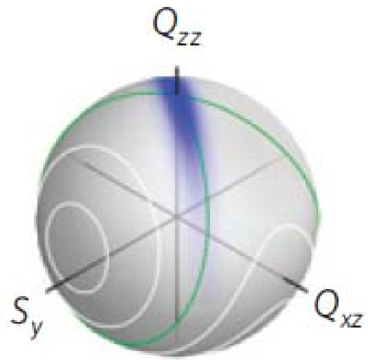


Quantum spin mixing

- Spin evolution when the mean field fails

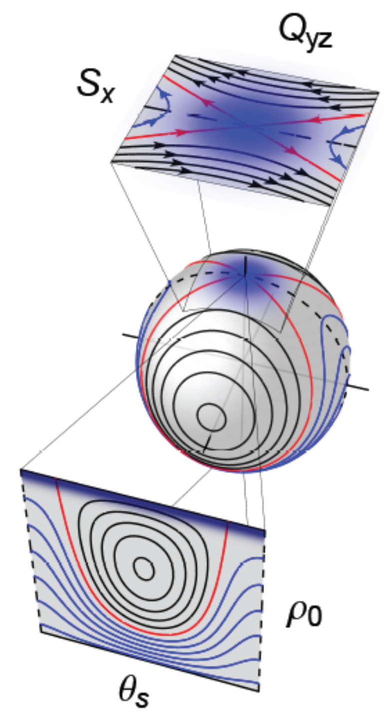


- Beyond the low depletion (vacuum) limit where squeezing is measured



Unstable hyperbolic fixed point

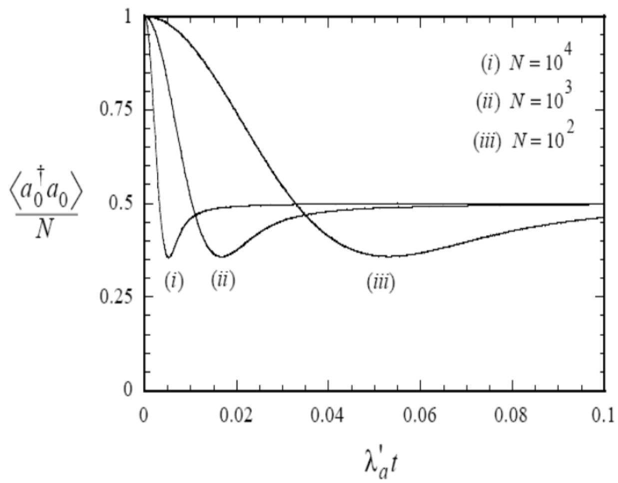
- The m_F = 0 state is a unstable equilibrium point
- In the mean-field limit, it is non-evolving
- Quantum fluctuations lead to evolution
  - In short term, squeezing is generated
  - In long term, highly non-Gaussian states are generated



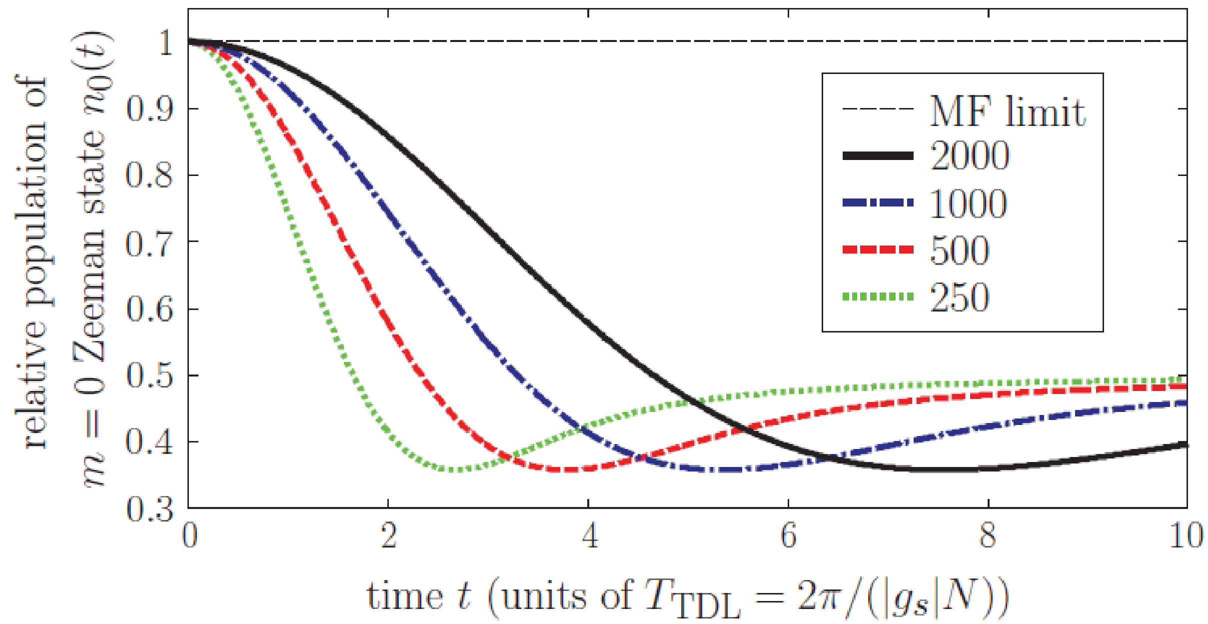
$$\mathcal{H} = \frac{c}{4}[x^2 - (1 - x^2)\cos\theta_s] + \frac{q}{2}(1 - x)$$

$$x \equiv 2\rho_0 - 1$$

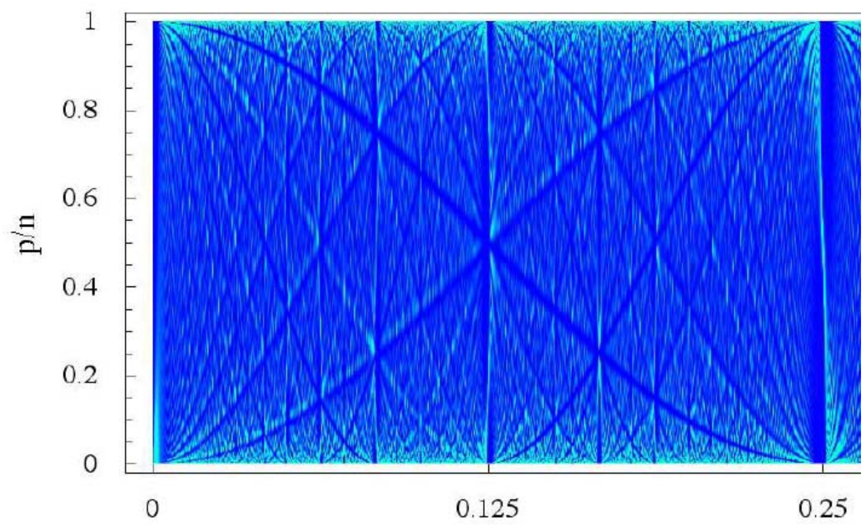
Theory



(i) $N = 10^4$
(ii) $N = 10^3$
(iii) $N = 10^2$
$\frac{\langle a_0^\dagger a_0 \rangle}{N}$
(i)
(ii)
(iii)
$\lambda'_a t$

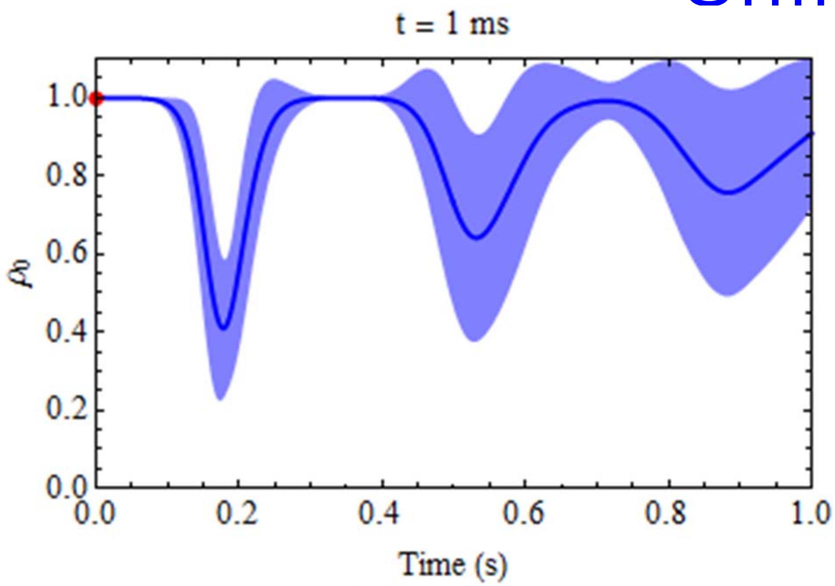


relative population of
$m = 0$ Zeeman state $n_0(t)$
MF limit
2000
1000
500
250
time t (units of $T_{\mathrm{TDL}} = 2\pi/(|g_s|N))$

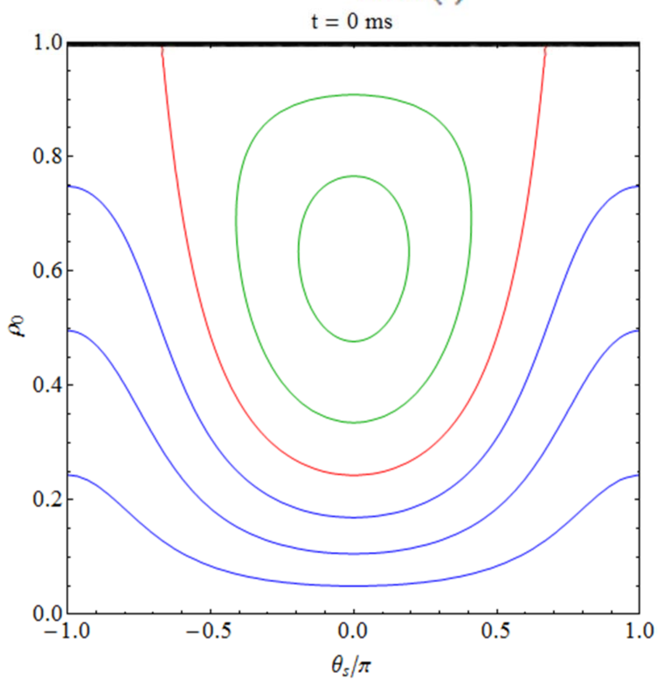


p/n

Simulation

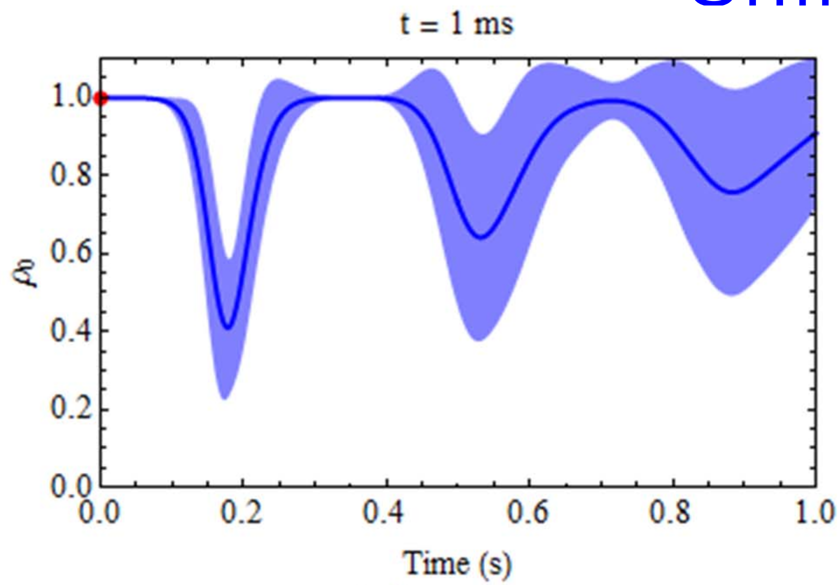


t = 1 ms
1.0
0.8
0.6
0.4
0.2
0.0
ρ_0
0.0
0.2
0.4
0.6
0.8
1.0
Time (s)

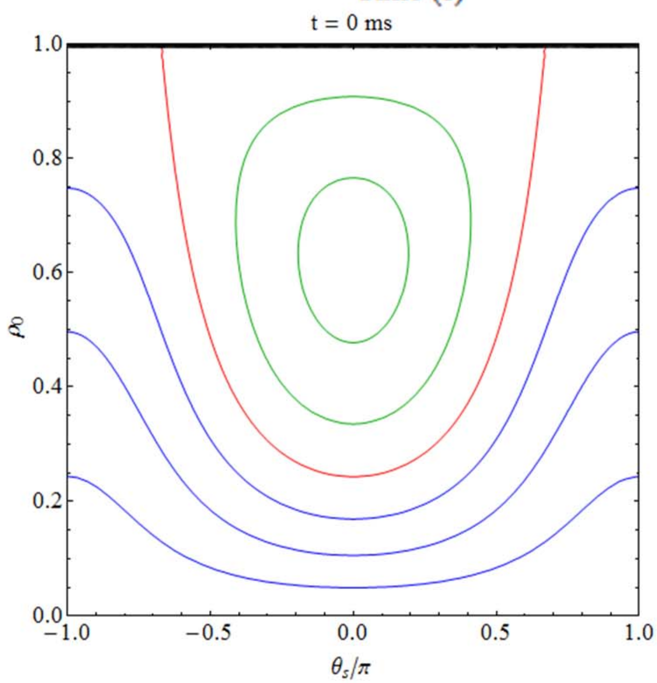


t = 0 ms
1.0
0.8
0.6
0.4
0.2
0.0
ρ_0
−1.0
−0.5
0.0
0.5
1.0
θ_s/π

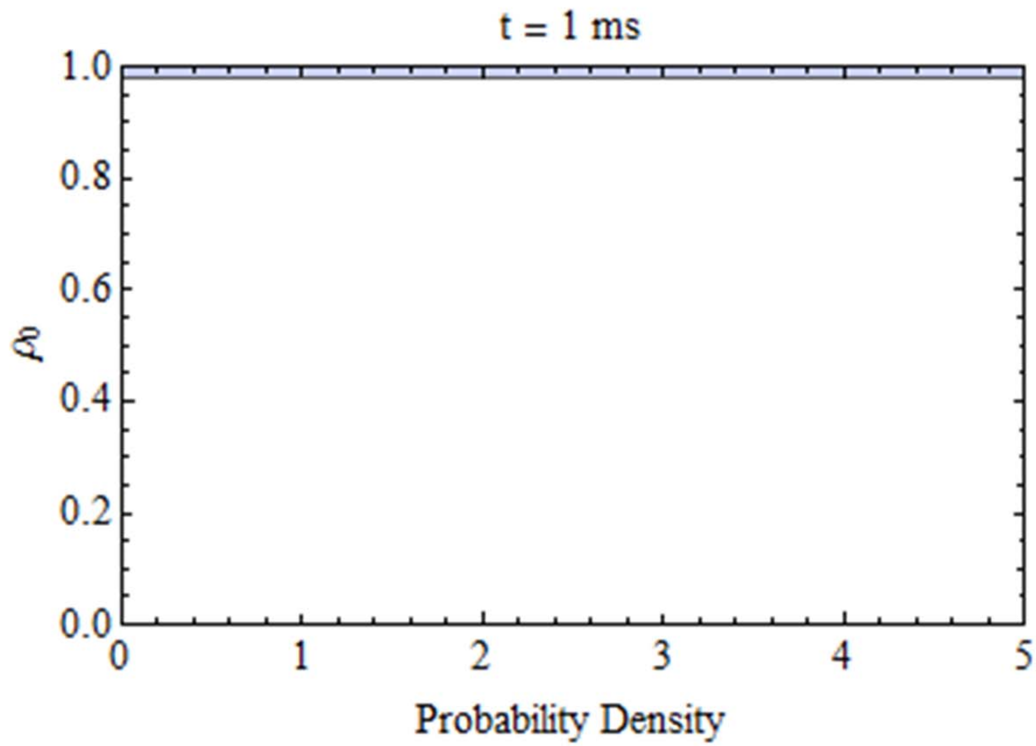
Simulation



t = 1 ms
1.0
0.8
0.6
0.4
0.2
0.0
ρ_0
0.0
0.2
0.4
0.6
0.8
1.0
Time (s)

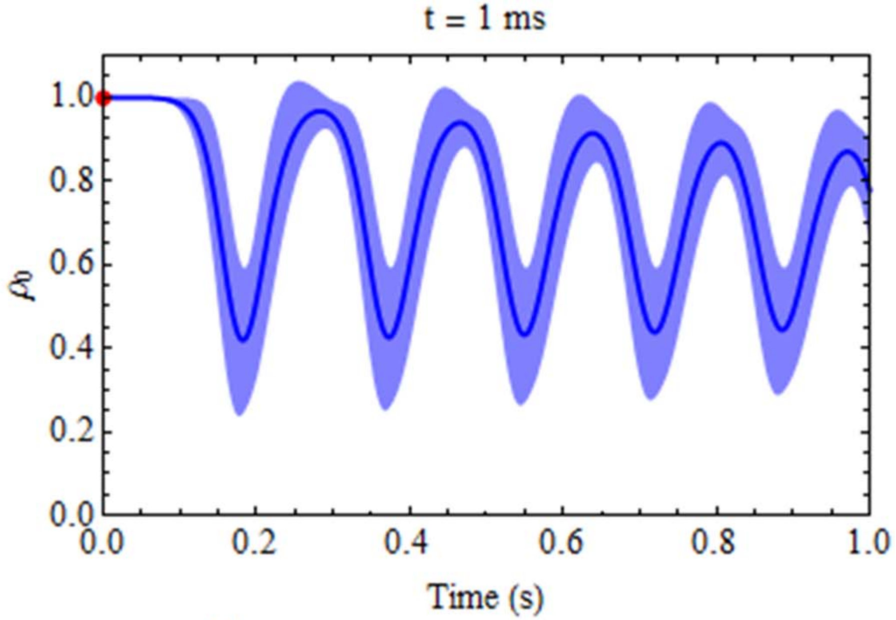


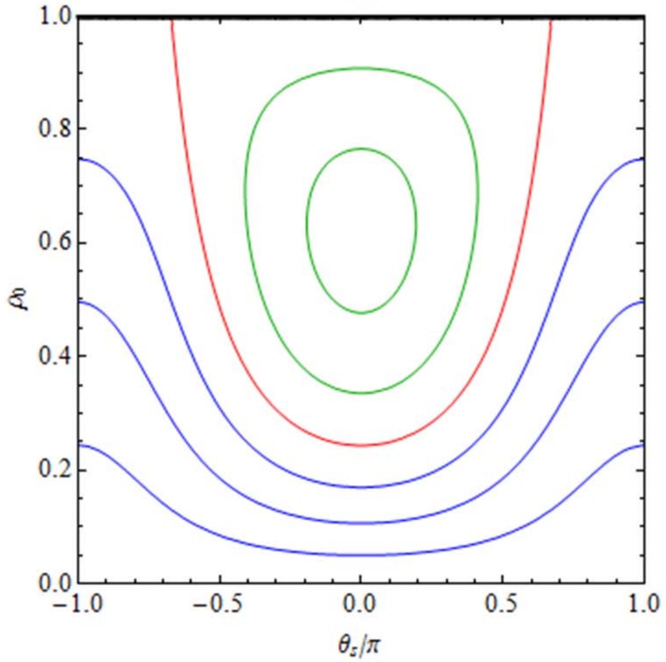
t = 0 ms
1.0
0.8
0.6
0.4
0.2
0.0
ρ_0
−1.0
−0.5
0.0
0.5
1.0
θ_s/π



t = 1 ms
1.0
0.8
0.6
0.4
0.2
0.0
ρ_0
0
1
2
3
4
5
Probability Density

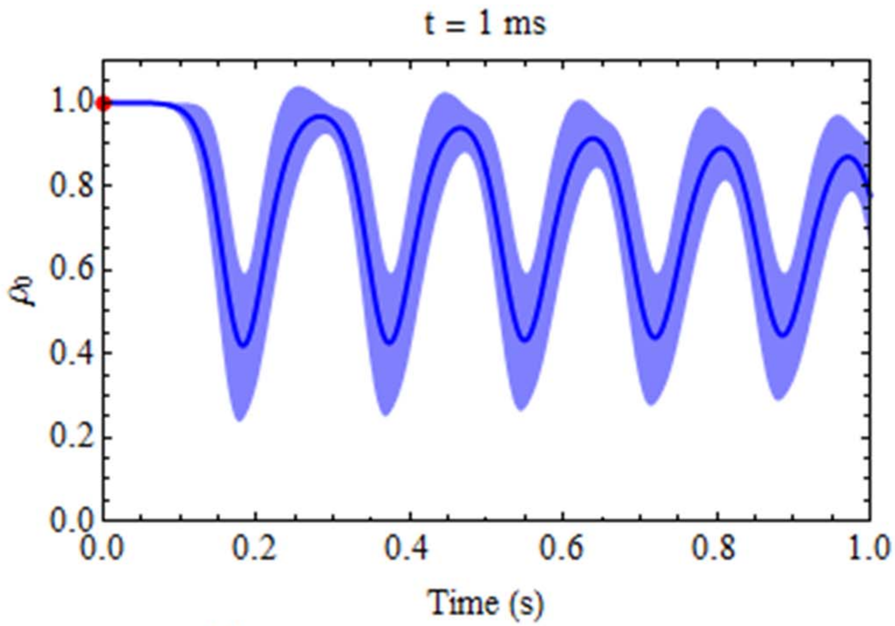
Simulation with loss



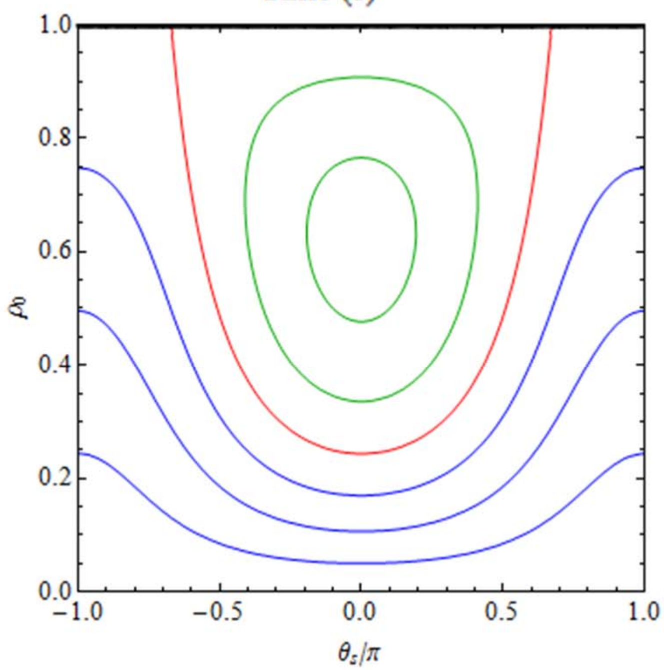


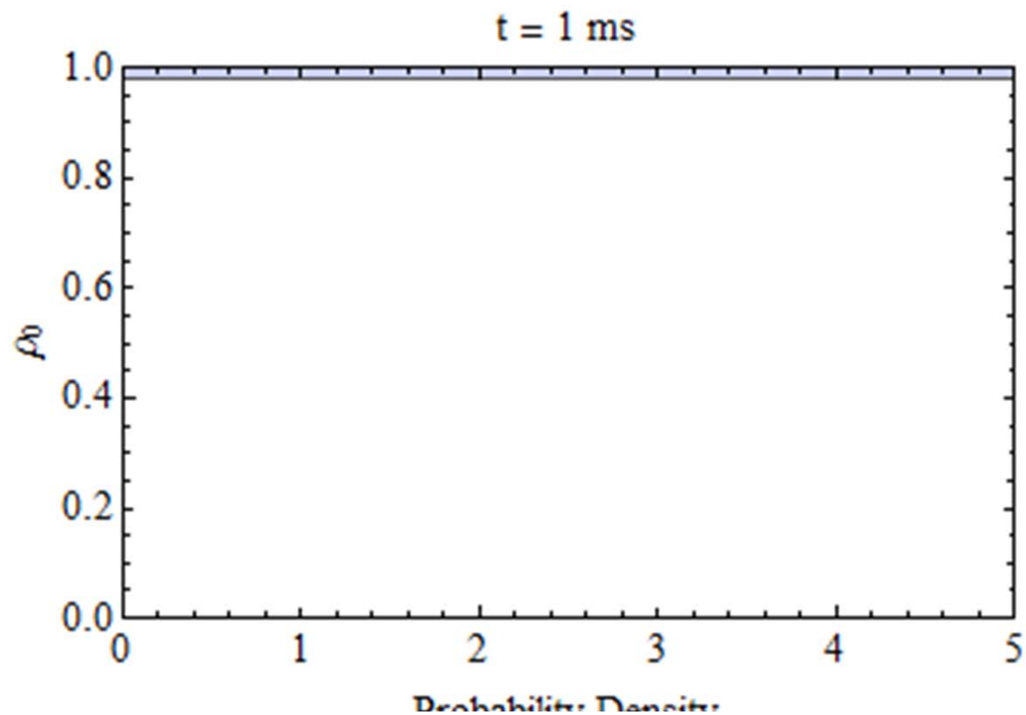
- Loss due to finite lifetime of condensate

Simulation with loss



- Loss due to finite lifetime of condensate

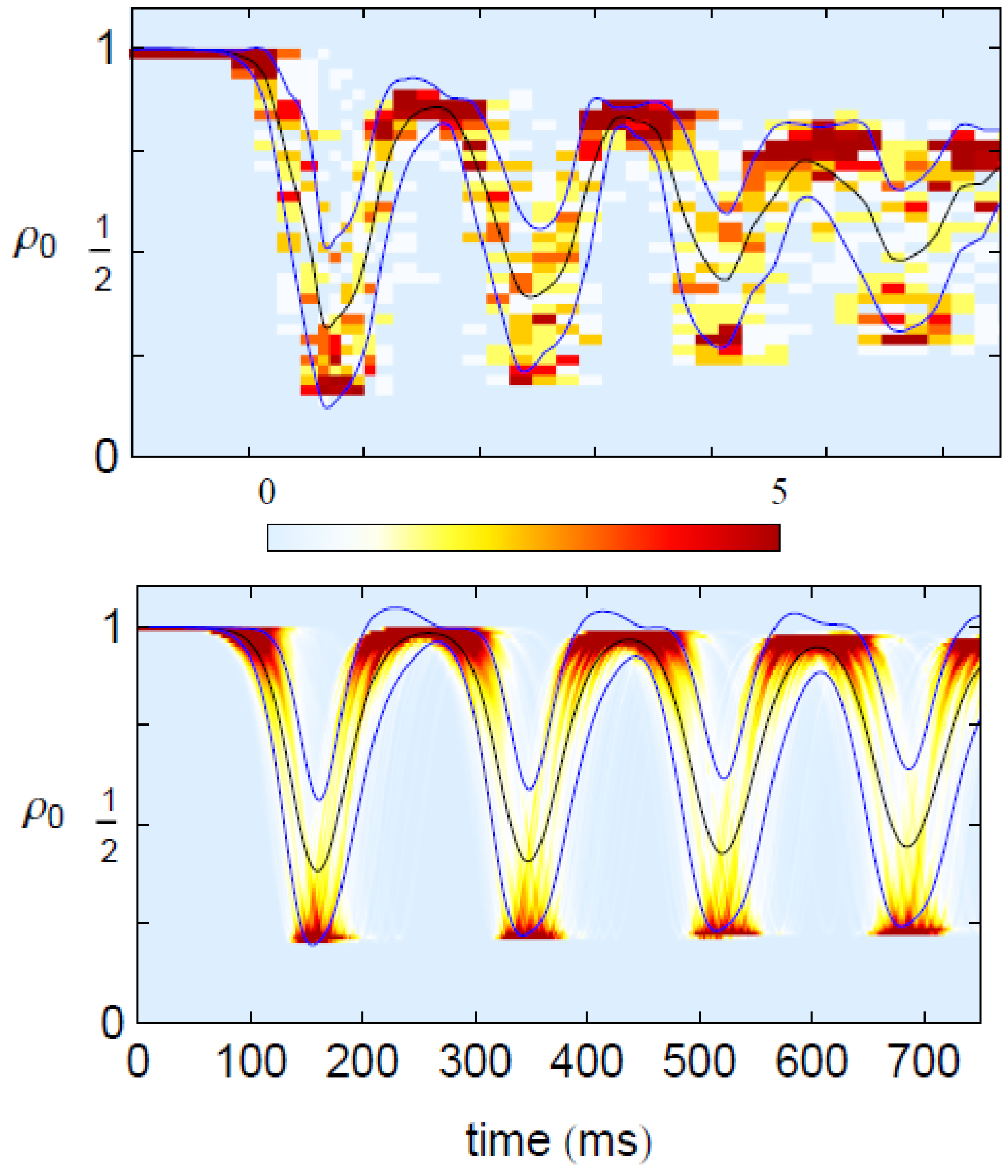




Experiment

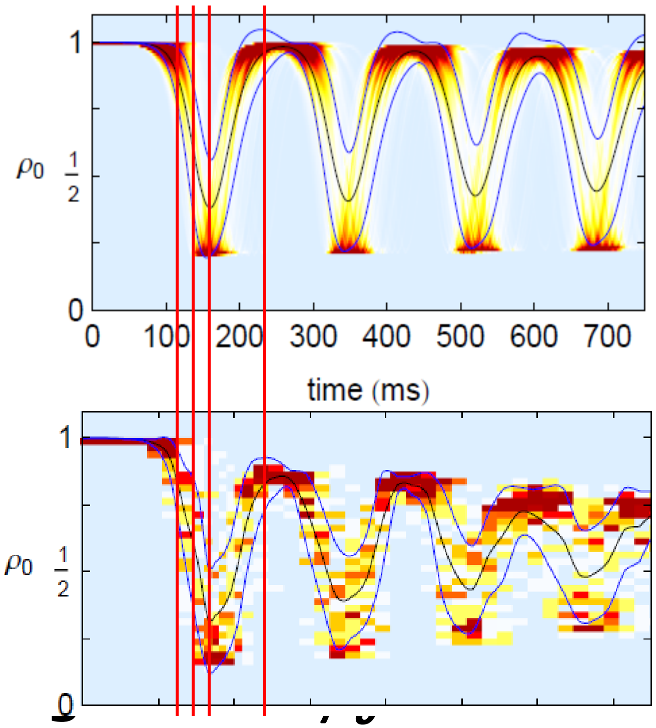
Measured probability distributions compared with theory

- Long initial pause
- Non-Gaussian distribution

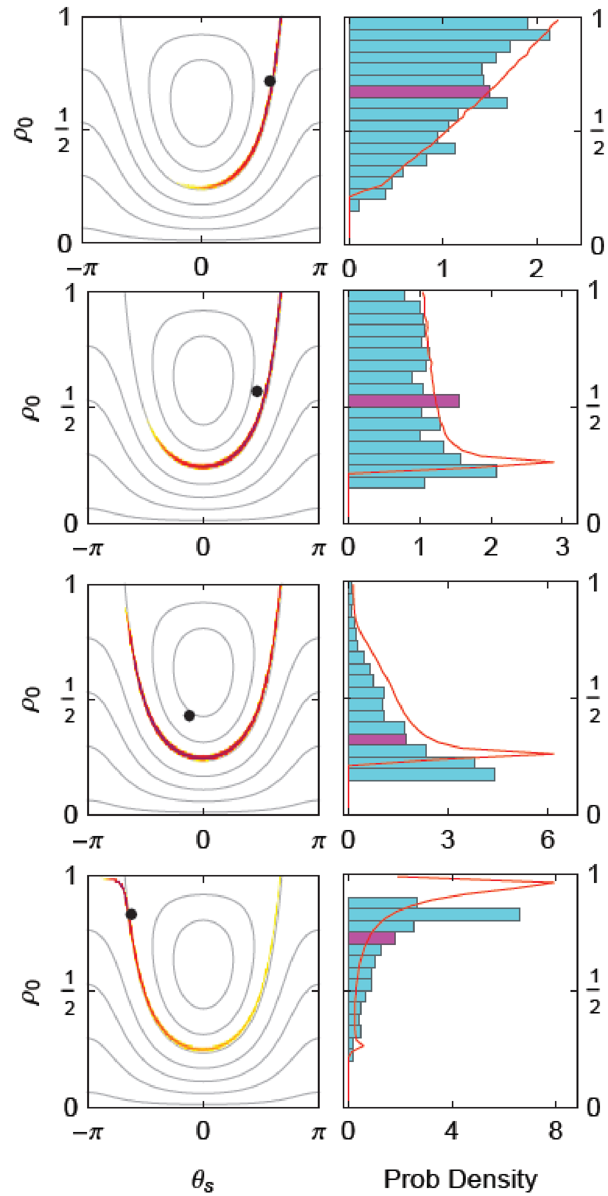


Non-Gaussian distributions

ρ_0 Histograms



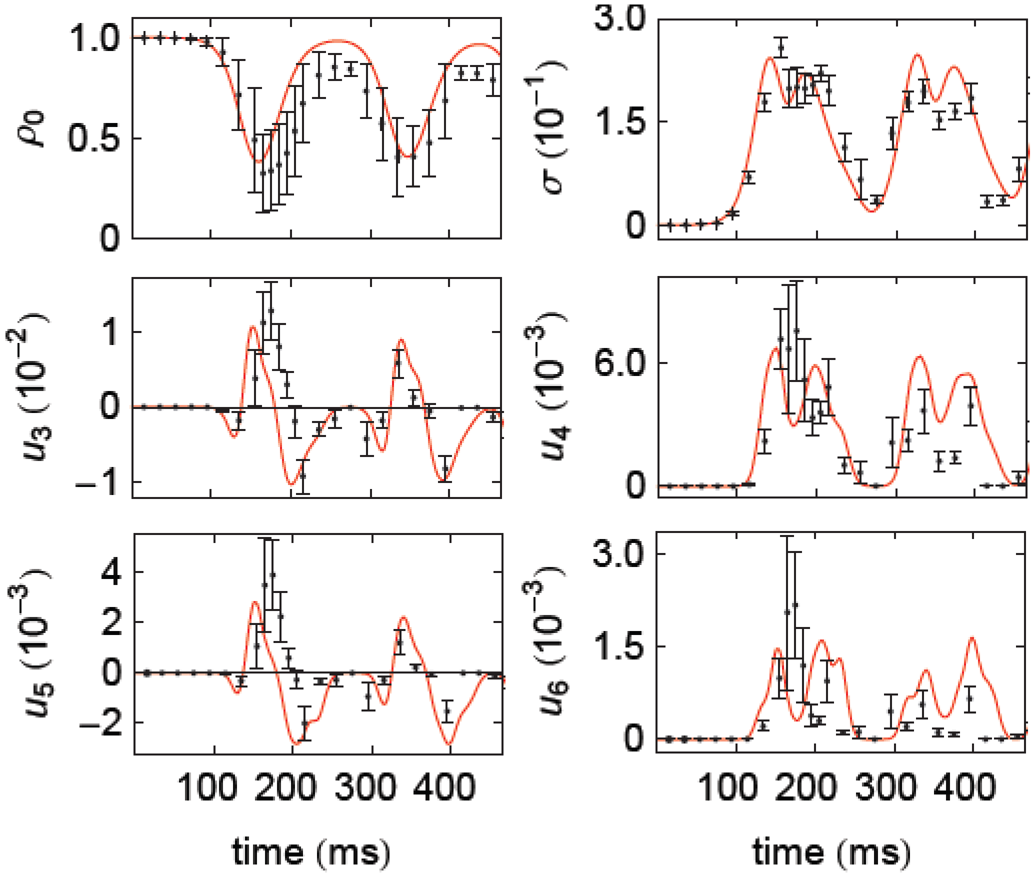
evolution times (130 ms, 140 ms, 170 ms, and 240 ms)



Central moments

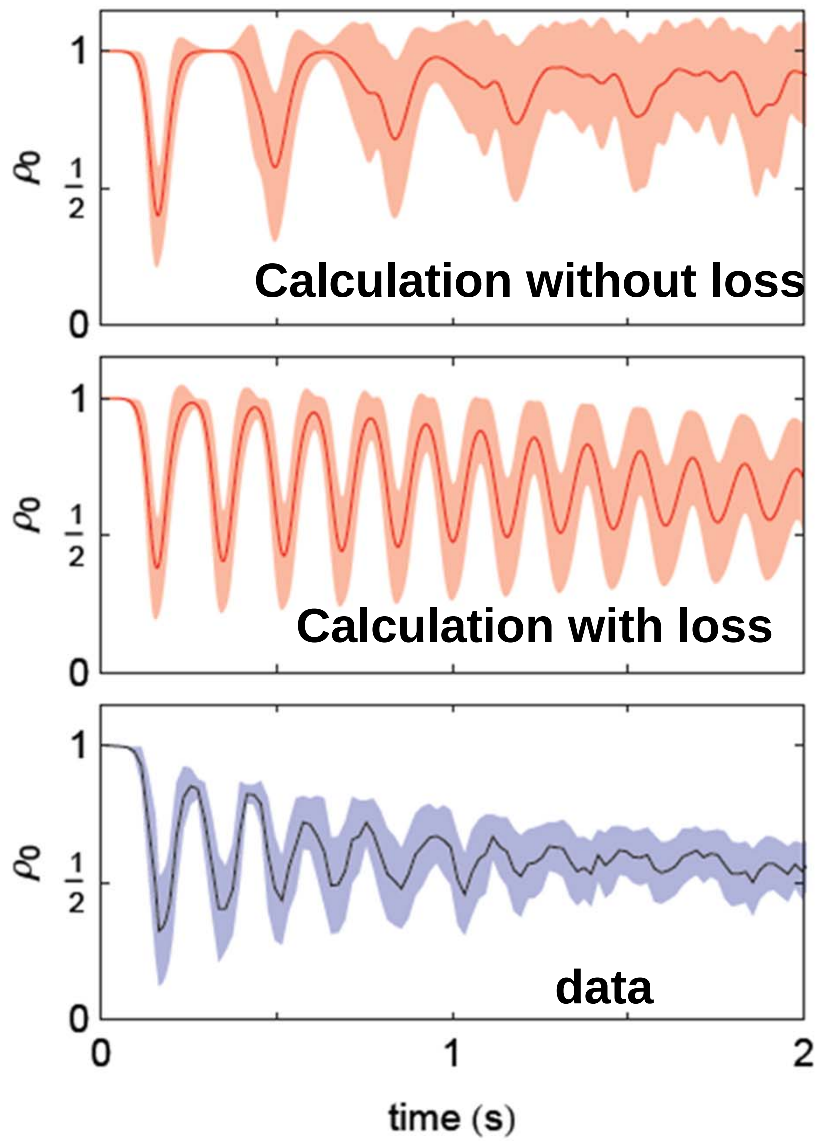
$$u_k = \left\langle \left(\rho - \bar{\rho}\right)^k \right\rangle$$

- Clear indication of non-Gaussian dynamics
- Population revival clearly visible in first four moments, less obvious in fifth and sixth.



Long term evolution

- Loss actually increases oscillation
  - Gain in mean-field coherence at expense of quantum correlations
- Long term evolution (t > 500 ms) shows continued oscillation with increased de-coherence in both the simulation and the data.
- The data does not revive as fully as the simulation, indicating other effects at longer time



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